On Functions of Bounded Mean Oscillation (BMO)

Let f be a function defined on a domain $\Omega \subseteq \mathbb{R}^n$.
Let $x \in \Omega$ and let $B(x,r) := \{y \in \mathbb{R}^n : \|y-x\| < r\} \subset \Omega$. Then

$$f_{ave}B(x,r) := \frac{1}{\|B(x,r)\|} \int\limits_{B(x,r)} f(y)\,dy =: \overline{f}_{x,r}.$$

♣ f is said to be of bounded mean oscillation over $\mathbb{R}^n$ written as

$$f \in BMO(\mathbb{R}^n) \text{ iff } \sup_{B(x,r)\subset\mathbb{R}^n} \frac{1}{\|B(x,r)\|} \int\limits_{B(x,r)} \mid f(y) - \overline{f}_{x,r} \mid dy < \infty.$$

Let us see few examples :

- Example 1 : $f(x) = k : \mathbb{R}^n \circlearrowleft$.

  Then

$$\overline{f}_{x,r} = \frac{1}{\|B(x,r)\|} \int\limits_{B(x,r)} f(y)\,dy = \frac{1}{\|B(x,r)\|}.k\|B(x,r)\| = k.$$

  Therefore

$$[f]_{BMO(\mathbb{R}^n)} = \sup_{B(x,r)\subset\mathbb{R}^n} \frac{1}{\|B(x,r)\|} \int\limits_{B(x,r)} \mid f(y) - \overline{f}_{x,r} \mid dy = 0.$$

  Conclusion: Constant functions are functions of Bounded Mean Oscillation with Mean Oscillation norm 0.

- Example 2. Consider the linear function $f(x) = x : \mathbb{R} \circlearrowleft$. Then

$$\overline{f}_{x,r} = \frac{1}{\| B(x,r) \|} \int\limits_{B(x,r)} f(y)\,dy = \frac{1}{\| B(x,r) \|} \int\limits_{B(x,r)} y dy = \frac{1}{\| B(x,r) \|} \frac{y^2}{2}\Big|_{B(x,r)}$$

$$= \frac{1}{2 \| B(x,r) \|} \left((x+r)^2 - (x-r)^2 \right) = \frac{4xr}{4r} = x.$$

  Find the Bonded Mean Oscillation Norm of $f : [f]_{BMO(\mathbb{R})}$.

  Solution:

$$[f]_{BMO(\mathbb{R})} = \sup_{B(x,r)} \frac{1}{\| B(x,r) \|} \int\limits_{B(x,r)} \mid f(y) - \overline{f}_{x,r} \mid dy$$

$$= \sup_{B(x,r)} \frac{1}{2r} \int\limits_{B(x,r)} | y-x | \, dy = \sup_{B(x,r)} \frac{1}{2r} \left[\int\limits_{x}^{x+r} (y-x) \, dy + \int\limits_{x-r}^{x} (x-y) \, dy \right]$$

$$= \sup_{B(x,r)} \frac{1}{2r} \left[r^2 \right] = \infty.$$

$\Rightarrow f \notin BMO\,(\mathbb{R})$.

- Problem: For what value(s) of $q \in \mathbb{R}$ is the function $f(x) = x^q \in BMO\,(\mathbb{R})$.