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Clifford analytic complete function systems for unbounded domains

Dejenie A. Lakew[1] and John Ryan[2,*]

[1] *Department of Mathematical Sciences and Technology, University of Arkansas at Pine Bluff, Pine Bluff, Arkansas 71603, U.S.A.*
[2] *Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701, U.S.A.*

SUMMARY

The main theme of this paper is to construct Clifford analytic-complete function systems in the generalized Bergman spaces: $B^p_{\mathrm{Cl}_n}(\Omega) := \ker D(\Omega) \cap L^p_{\mathrm{Cl}_n}(\Omega)$, and $B^{p,2}_{\mathrm{Cl}_n}(\Omega) := \ker \triangle(\Omega) \cap L^p_{\mathrm{Cl}_n}(\Omega)$. These systems are used to approximate null solutions of elliptic partial differential equations of the Dirac and Laplace operators over an unbounded domain Ω in $\mathbb{R}^n$. Copyright © 2002 John Wiley & Sons, Ltd.

1. INTRODUCTION

Boundary value problems of linear and non-linear partial differential equations have long been studied through the techniques and theory of integro-differential operators and approximation theory. The basic ideas of this paper are motivated by works of Gürlebeck and Sprößig, see for instance References [1,2]. There a quaternionic calculus in particular and Clifford calculus in general is developed for the treatment of several kinds of boundary value problems by both analytical and approximation techniques. The domains they work on are bounded and Liapunov. In this paper, we work on domains which are unbounded and have C^2 boundaries, though our results can be extended to unbounded Liapunov domains, so the boundary would then be C^1 with Hölder continuous normal vector.

Clifford analysis over unbounded domains has been studied by many authors using different techniques, see for instance References [3–8,15]. In Reference [4], it is clearly indicated that a similar function theory can be developed to those developed in References [1,2] over unbounded domains to study Clifford analytic functions. The authors there prove the existence of a Cauchy transform as well as a Cauchy integral formula in Hardy spaces and Hölder spaces. The problem in passing directly to unbounded domains is that the Cauchy kernel does not decay fast enough at infinity. The usual T-transform or Cauchy transform is not absolutely convergent in the usual function spaces. The function theory developed in References [4,5] and elsewhere helps to amend this problem. In Reference [4], the amendment is done by

Correspondence to: J. Ryan, Department of Mathematical Sciences, University of Arkansas, Fayetteville, Arkansas 72701, U.S.A.

adding an extra term to the Cauchy kernel. This idea is not new, see for instance Reference [6]. In Reference [5], the authors first assume that the complement of Ω contains a non-empty open set and then introduce the following kernel:

$$\Psi^z(x,y) = \frac{1}{\omega_n}\left(\frac{(x-y)}{\|x-y\|^n} - \frac{(x-z)}{\|x-z\|^n}\right)$$

where, z is an arbitrary but fixed point lying in an open set in the complement of Ω.

In this paper, we extend some of the results from References [1,2] to domains which are unbounded and work in a general Clifford analysis setting using the function theory developed in References [4,5] with this modified kernel. We present a decomposition of a harmonic function as a sum of a monogenic function and a Cauchy transform of another monogenic function. We construct Clifford analytic-complete function systems in the null spaces of the differential operators which are p integrable over Ω and get approximation results.

2. PRELIMINARIES

Let Cl_n denote the real 2^n-dimensional Clifford algebra generated from R^n under the multiplication rule $x^2 = -\|x\|^2$ for each $x \in R^n$. It should be noted that in this case we are assuming that R^n is embedded in Cl_n. Also each non-zero vector $x \in R^n$ has a multiplicative inverse, $-x/\|x\|^2$. Up to a sign this is the Kelvin inverse of x. If $e_1,\ldots,e_n$ is an orthonormal basis of $\mathbb{R}^n$, then from the previous multiplication, we have: $e_ie_j + e_je_i = -2\delta_{i,j}e_0$, for $i,j=1,\ldots,n$ where $\delta_{i,j}$ is the Kronecker delta, and $e_0, = 1$, is the identity element of the algebra. Thus, each $a \in \mathrm{Cl}_n$ can be written as $\sum_{A\subset\{1,\ldots,n\}} a_Ae_A$ with real coefficients a_A. Moreover the norm of an element a can now be defined to be $\|a\| = (\sum_A a_A^2)^{1/2}$.

The function spaces considered in this paper include Sobolev and Slobedeckij spaces.

Let ∂^m denote $\partial^{r_1}/\partial x_1^{r_1}\cdots\partial^{r_n}/\partial x_n^{r_n}$ with $r_1+\cdots+r_n = m$ and let $\phi \in C^\infty(\Omega)$. Then the support of ϕ, denoted by $\operatorname{supp}\phi$ is the closure of the set, $\{x \in \Omega\colon \phi(x) \neq 0\}$. Then when Ω is bounded $C_0^\infty(\Omega) := \{\phi \in C^\infty(\mathbb{R}^n)\colon \operatorname{supp}\phi \subset K^{\text{compact}} \subset \Omega\}$.

Definition 1
A locally integrable function f defined on Ω has a locally integrable weak or distributional partial derivative of order r, denoted by $\partial^r f$ if.-

$$\int_\Omega f(x)\partial^r\phi(x)\,\mathrm{d}x^n = (-1)^r \int_\Omega \partial^r f(x)\phi(x)\,\mathrm{d}x^n$$

for all $\phi \in C_0^\infty(\Omega)$.

Definition 2
The Sobolev space $W^{p,m}(\Omega)$ for $1<p<\infty$ is defined to be the Banach space $[\{f \in L^p(\Omega)\colon \partial^m f \in L^p(\Omega)\}$ with norm

$$\|f\|_{p,m} := \left(\sum_{0\leqslant r\leqslant m} \|\partial^r f\|_p^p\right)^{1/p}$$

The space $W^{0,p,m}(\Omega)$ is the space $\{f \in W^{p,m}(\Omega)\colon \mathrm{tr}_\Omega f = 0\}$. Note that $W^{p,0} = L^p(\Omega)$.

When $0 < s \neq$ integer, and $1 < p < \infty$, then the function space $W^{p,s}(\Omega)$ is called a Slobedeckij space. For detailed information on such spaces, see Reference [9]. The Slobedeckij spaces are closely related to the investigation of boundary values of functions which belong to some Sobolev spaces. In our case these spaces are traces or boundary values of functions from some Sobolev spaces. The trace operator is the one which is continuous and has the mapping property: $\mathrm{tr}_\Omega : W^{p,m}(\Omega) \to W^{p,m-1/p}(\partial\Omega)$.

The Clifford analytic analogue of the Cauchy–Riemann operator is called the Dirac operator and it is written as $D = \sum_{j=1}^{n} e_j\,(\partial/\partial x_j)$. One may immediately see that the Laplacian in $\mathbb{R}^n$ can be factored as $\triangle = -D^2$. A Cl_n-valued function f defined on Ω is called left monogenic if $Df(x) = 0$ for every $x \in \Omega$ and right monogenic, if $fD(x) = 0$ for every $x \in \Omega$. Here, $fD = \sum_{j=1}^{n} (\partial f/\partial x_j) e_j$. One may readily deduce that monogenic functions are harmonic. See Reference [10] for details.

An important example of a function that is both a left and right monogenic function is given by $E(x) = -x/\|x\|^n$ defined on $\mathbb{R}^n \backslash \{0\}$ and it is a generalization of the Cauchy kernel from one variable complex analysis. See Reference [10].

Among the many properties of the Dirac operator D the one which we need most and in fact which makes Clifford analysis more interesting is its right invertibility over Sobolev spaces. Over a bounded domain $\Omega \subset \mathbb{R}^n$, its right inverse is given by the Teodorescu integral operator commonly known as the T-transform and defined by $T_\Omega(f)(x) = 1/\omega_n \int_\Omega E(x-y) f(y)\,\mathrm{d}y^n$. Over unbounded domains it is given by the generalized Teodorescu-transform defined by $\tilde{T}_\Omega(f)(x) = \int_\Omega \Psi^z(x,y) f(x)\,\mathrm{d}y^n$, where, in the above two integral operators E is the ordinary Cauchy-kernel while Ψ^z is the modified Cauchy-kernel mentioned in the introduction. The latter operator is defined in Reference [5].

Lemma 1 (*Gürlebeck and Sprößig* [*1*])
Let $f \in C^1(\Omega) \cap C(\bar{\Omega})$, where Ω is a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary. Then $DT_\Omega f = f$. That is $D_R^{-1} = T_\Omega$, where, D_R^{-1} denotes the right inverse of D.

The reason why the Cauchy kernel needs to be modified when working on unbounded domains is given by the following result.

Lemma 2
Let Ω be an unbounded domain in $\mathbb{R}^n$ and suppose $\varepsilon > 0$. Let $\Omega_\varepsilon = \Omega \backslash B(y,\varepsilon)$, where $B(y,\varepsilon)$ is a ball centred at y and radius ε. Then, $E(x-y) \in L^p(\Omega_\varepsilon)$, for $n/(n-1) < p < \infty$

Proof

$$\int_{\Omega_\varepsilon} \|E(x-y)\|^p\,\mathrm{d}y^n = \int_{\Omega_\varepsilon} \frac{1}{\|x-y\|^{p(n-1)}}\,\mathrm{d}y^n \leqslant C \int_\varepsilon^\infty r^{p(1-n)+n-1}\,\mathrm{d}r < \infty$$

if $p > n/(n-1)$. □

Thus, if $n = 3$ we see that $E(x,y) \in L^p(\Omega_\varepsilon)$ for $\frac{3}{2} < p < \infty$. In particular it is square integrable. But for $n = 2$, $E(x,y)$ is not square integrable on Ω_ε. This dependency of the index p on the dimension of the Euclidean space caused by the behaviour of the Cauchy kernel at infinity is avoided by either working on weighted spaces [3] or by using Möbius transformations [11] or by perturbing the T_Ω transform by adding a smooth monogenic term whose singularity lies outside of the domain under consideration [4,5,12]. For our work our choice is the last one. The following lemma is instrumental in establishing the p integrability of the modified kernel Ψ^z for any $p>1$ over unbounded domains.

Lemma 3 (*Kähler* [*12*])
Suppose $\|x\|>2\max(\|y\|,\|z\|)$. Then

$$\|E(x-y)-E(x-z)\| \leqslant C(n)\,\frac{\|y-z\|+\max(\|y\|,\|z\|)}{\|x\|^n}$$

where $C(n)=2^{n+1}(n-1)$ is a constant that depends on the dimension n.

This lemma guarantees the absolute convergence of a modified T_Ω transform for unbounded domains. The following fundamental result follows immediately from the above lemma.

Proposition 1
Let Ω be an unbounded domain whose complement contains a non-empty open set. Let z be an arbitrary but fixed point lying in that complementary open set, and let $\Omega_\varepsilon=\Omega\backslash B(y,\varepsilon)$ for $\varepsilon>0$. Then $\Psi^z\in L^p(\Omega_\varepsilon)$ for $1<p<\infty$.

Proof
Let $\Omega_r=\Omega\cap B(0,r)$, where $B(0,r)$ is a big ball centred at 0 and radius r so that x, y and z are contained in it. Then $\int_{\Omega_\varepsilon}\|\Psi^z(x,y)\|^p\,\mathrm{d}\Omega_{\varepsilon,x}\leqslant\lim_{R\to\infty}C(n,p)\int_\varepsilon^R r^{-np+n-1}\,\mathrm{d}r<\infty$ for $p>1$. Hence, the generalized Teodorescu transform is absolutely convergent on $W^k_{q,\mathrm{Cl}_{0,n}}(\Omega)$ for $q>1$ such that $p^{-1}+q^{-1}=1$ and $k\in N\cup\{0\}$. □

Proposition 2
The $\tilde{T}_\Omega$ transform is the right inverse of the Dirac operator over $W^{q,k}_{\mathrm{Cl}_n}(\Omega)$ where Ω is an unbounded domain in R^n, with C^2 boundary, for $k\in N\cup\{0\}$ and $q>1$.

Proof
Again, let r be large enough so that $y\in B(0,r)$, $y\in\Omega$. Then for $z\in\mathrm{cl}(\Omega)^c$ and $\Omega_r=\Omega\cap B(0,r)$, we have

$$\int_\Omega \Psi^z(x,y)f(x)\,\mathrm{d}\Omega_x=\int_{\Omega_r}\Psi^z(x,y)f(x)\,\mathrm{d}\Omega_x+\int_{\Omega\backslash\Omega_r}\Psi^z(x,y)f(x)\,\mathrm{d}\Omega_x$$

Here $\mathrm{cl}(\Omega)$ is the closure of Ω. From the Borel–Pompeiu formula, we have $D\Psi^z=0$ over this domain. That is, the second summand of the above integral equation is monogenic. And from the case for bounded domains, we have that $D_y\int_{\Omega_R}\Psi^z(x,y)u(x)\,\mathrm{d}\Omega_x=u(y)$. □

Proposition 3
For $1<p<\infty$, $k=0,1,2,\ldots$, and Ω an unbounded domain with C^2 boundary, the operator $\tilde{T}_\Omega: W^{p,k}_{\mathrm{Cl}_n}(\Omega)\to W^{p,k+1}_{\mathrm{Cl}_n}(\Omega)$ is a continuous mapping.

Proof
This is the same as the case $p=2$ in Reference [1]. □

From [9] one may also determine:

Proposition 4
For $1<p<\infty$, and Ω an unbounded domain with C^2 boundary, the operator $\tilde{T}_\Omega : W^{p,-1}_{\mathrm{Cl}_n}(\Omega)\to L^p_{\mathrm{Cl}_n}(\Omega)$ is a bounded operator.

Definition 3
For $1<p<\infty$, the set $\{f:\Omega\to\mathrm{Cl}_n$: f is left monogenic and $f\in L^p_{\mathrm{Cl}_n}(\Omega)\}$ is called the Bergman p-space for Ω and it is denoted by $B^p_{\mathrm{Cl}_n}(\Omega)$.

Proposition 5
Let Ω and Ω_L be unbounded domains in R^n with C^2 boundaries and such that $\Omega_L\supset\bar{\Omega}$ with $\Sigma_L=\partial\Omega_L$ and $\sum=\partial\Omega$ be C^2 hypersurfaces. Let z be an arbitrary but fixed point in $R^n\backslash\bar{\Omega}$ and $\{x_m$: $m\in N\}$ be a dense subset of $\sum_L$. Then for each $m\in N$ the function $\Psi^z_m(x)=\Psi^z(x,x_m)=E(x-x_m)-E(x-z)$ is in $B^p_{Cl_{0,n}}(\Omega)$ for $1<p<\infty$.

Proof
Clearly $D\Psi^z_n(x)=0$. And the p-integrability follows from Proposition 1 above. □

Proposition 6
Suppose that $f\in B^p_{\mathrm{Cl}_n}(\Omega)$ for some $p\in(1,\infty)$ then for each point $y\in\Omega$ and each C^2 hypersurface Γ bounding a subdomain Ω' of Ω and with $y\in\Omega'$ we have $f(y)=1/\omega_n\int_\Gamma\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$.

Proof
We need only consider the cases where Ω' is unbounded. Consider the closed ball $D(y,R)=\{x\in R^n$: $\|x-y\|\leqslant R\}$. Let $\Gamma_1(R)=\partial D(y,R)\cap(\Omega'\cup\Gamma)$ and $\Gamma_2(R)=\partial(\Omega'\backslash D(y,R)$. Clearly $f(y)=1/\omega_n\int_{\Gamma_1(R)}\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$. As $f\in L^p(\Omega)$ then $(\int_{\Omega'\backslash D(y,R)}\|f\|^p\,\mathrm{d}x^n)^{1/p}<\infty$. So for a given $\varepsilon>0$ we can find an $R(\varepsilon)$ such that $(\int_{\Omega'\backslash D(y,R)}\|f\|^p\,\mathrm{d}x^n)^{1/p}<\varepsilon$ for each $R>R(\varepsilon)$. Let us denote the set $\Omega'\backslash D(y,R)$ by $\Omega'(R)$. The subset $X(R)$ of $\Omega'(R)$ for which $\|f(x)\|>\varepsilon$ has to have measure less than $\varepsilon^{1/p}$ whenever $\varepsilon<1$. So $\|f\|<1$ on $\Omega'(R)\backslash X(R)$. So on any subdomain $\Omega''(R)$ of $\Omega'(R)\backslash X(R)$ with C^2 boundary $\Gamma(R)$ the integral $\int_{\Gamma(R)}\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$ vanishes. The result follows on letting R tend towards infinity. □

Using the modified Cauchy kernel Ψ^z, we introduce two integral operators on the function space $L^p_{\mathrm{Cl}_n}(\sum)$. For $1<p<\infty$ the Cauchy-type boundary integral operator is defined to be

$$\tilde{F}_{\sum}f(y):=\int_{\sum}\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

where $y\notin\sum$. The singular integral operator of Cauchy-type is formally defined to be

$$\tilde{S}_{\sum}f(y):=2\int_{\sum}\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

where $n(x)$ is the outward pointing unit normal vector to $\sum$ at the point x and $y\in\sum$. Here, $\tilde{S}_{\sum}$ is understood in terms of the Cauchy principal value. From the celebrated results of Coifman *et al.* [14], one can show that $\tilde{S}_{\sum}$ is a bounded mapping of $L^p(\sum)$ for $1<p<\infty$ with the weaker assumption that $\sum$ is a Lipschitz surface. We will only use the fact that $\sum$ is C^2 or better. The Coifman–McIntosh–Meyer Theorem tells us that the operator

$S_{\sum} f(y) = 1/\omega_n\, p.v. \int_{\sum} E(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)$ is L^p bounded and hence, since the extra term that is added to the kernel to form the modified kernel has no singularity in $\sum$ we have the L^p norm of the integral $\int_{\sum} E(x-z)n(x)f(x)\,\mathrm{d}\sigma(x)$, where z is a fixed but arbitrary point outside of $\sum$, is dominated by the L^p norm of the integral $p.v. \int_{\sum} E(x-y)n(x)f(x)\,\mathrm{d}\sigma(x)$. See Reference [15] for closely related results.

Theorem 1 (*Kähler* [*12*]) (*The Borel–Pompeiu formula over unbounded domains*)
Let $f \in W^{p,k}_{\mathrm{Cl}_n}(\Omega)$, $1 < p < \infty$, $k \in N \cup \{0\}$ and Ω be an unbounded domain in R^n and with C^2 boundary. Then $f = \tilde{F}_{\sum} f + \tilde{T}_{\Omega} Df$ for each $x \in \Omega$.

Using the Clifford analysis version of the Cauchy integral formula described in Reference [10] and elsewhere we have:

Theorem 2 (*Gürlebeck and Sprößig* [*2*]*Luzin*)
Let $f \in C^1(\Omega) \cap C(\mathrm{cl}(\Omega))$ and f is monogenic in Ω. Let also $\Gamma \subset \Omega$ be a $(n-1)$-dimensional manifold with $f(x) = 0$ on Γ. Then $f \equiv 0$ on $\mathrm{cl}(\Omega)$.

Proposition 7 (*see also Bernstein* [*3*]*, Gurlebeck et al.* [*5*]*, Kähler* [*12*])
Let Ω, p and k be as above. Then the operator

$$\tilde{F}_{\sum} : W^{p,k-1/p}_{\mathrm{Cl}_n}\left(\sum\right) \to W^{p,k}_{\mathrm{Cl}_n}(\Omega) \cap \ker D(\Omega)$$

is a continuous operator.

Proof
Let $f \in W^{p,k-11/p}_{\mathrm{Cl}_n}(\sum)$. Then there exists a Cl_n-valued extension $g \in W^{p,k}_{\mathrm{Cl}_n}(\Omega)$ with $\mathrm{tr}_{\sum} g = f$. Using the Borel–Pompeiu formula, we get $\tilde{F}_{\sum} f + \tilde{T}_{\Omega} Dg = g$. Then since $\tilde{T}_{\Omega} : W^{p,k}_{\mathrm{Cl}_n}(\Omega) \to W^{p,k+1}_{\mathrm{Cl}_n}$ (Ω) and $D : W^{p,k}_{\mathrm{Cl}_n}(\Omega) \to W^{p,k-1}_{\mathrm{Cl}_n}(\Omega)$ are continuous, we see that the operator $I_{\Omega} - \tilde{T}_{\Omega} D$ is continuous from $W^{p,k}_{\mathrm{Cl}_n}(\Omega)$ to $W^{p,k}_{\mathrm{Cl}_n}(\Omega)$ where, I_{Ω} is the identity operator over Ω. □

Also, the Plemelj formulae are obtained just by looking at the traces of $\tilde{F}_{\sum} f$:

$$\tilde{P}_{\sum} f(x) := \lim_{n.t.\xi \to x} \tilde{F}_{\sum} f(\xi) = \tfrac{1}{2}(f(x) + \tilde{S}_{\sum} f(x))$$
$$\tilde{Q}_{\sum} f(x) := \lim_{n.t.\xi \to x} -\tilde{F}_{\sum} f(\xi) = -\tfrac{1}{2}(f(x) - \tilde{S}_{\sum} f(x))$$

The Plemelj operators that we just introduced are extendible to the usual function spaces $L^p_{\mathrm{Cl}_n}(\Omega)$ and $W^{p,k}_{\mathrm{Cl}_n}(\Omega)$ for $k \in N$, as Hölder continuous functions of compact support are dense in the latter spaces.

3. COMPLETE SPACES

In this section, we will see a unique representation of a Clifford-valued harmonic function $f : \Omega \to \mathrm{Cl}_n$ as a sum of a monogenic function and a $\tilde{T}_{\Omega}$ transform of another monogenic function, and some decomposition results of p-integrable functions over unbounded domains in R^n. Also we search for special left-monogenic functions which are dense in the generalized Bergman p space $B^p_{\mathrm{Cl}_n}(\Omega)$. We begin with the following result which we will need later.

Lemma 4

Let $1<p<\infty$ and $f\in\ker\triangle(\Omega)\cap W^{p,k}_{\mathrm{Cl}_n}(\Omega)$, $k\in N\cup\{0\}$. Then there exist unique functions $f_i\in\ker D(\Omega)\cap W^{p,k+1-i}_{\mathrm{Cl}_n}(\Omega)$, $i=1,2$ such that $f=f_1+\tilde{T}_\Omega f_2$.

Proof

From the Borel–Pompeiu formula, we have $f=\tilde{F}_{\sum}f+\tilde{T}_\Omega Df$ in Ω, and also $\tilde{F}_{\sum}f\in W^{p,k}_{\mathrm{Cl}_n}(\Omega)\cap\ker D(\Omega)$. Set $f_1=\tilde{F}_{\sum}f$ and $f_2=Df$. Then $f_2\in W^{p,k-1}_{\mathrm{Cl}_n}(\Omega)\cap\ker D(\Omega)$, and $f_1\in\ker D(\Omega)\cap W^{p,k}_{\mathrm{Cl}_n}(\Omega)$.

For the uniqueness, suppose $f=g_1+\tilde{T}_\Omega g_2$. Then $Df=g_2=f_2$ and $\tilde{T}_\Omega g_2-\tilde{T}_\Omega f_2=\tilde{T}_\Omega(g_2-f_2)=0$. □

Theorem 3 (*Kähler* [*12*])

Let $1<p<\infty$, and Ω be an unbounded domain with C^2 boundary. Then $L^p_{\mathrm{Cl}_n}(\Omega)$ has a direct decomposition.

$$L^p_{\mathrm{Cl}_n}(\Omega)=B^p_{\mathrm{Cl}_n}(\Omega)\oplus D(W^{0,p,1}_{\mathrm{Cl}_{0,n}}(\Omega))$$

where $\oplus$ here is a direct sum. It is an orthogonal sum when $p=2$.

This direct decomposition of $L^p_{\mathrm{Cl}_n}(\Omega)$ gives us projections:

$$P:L^p_{\mathrm{Cl}_n}(\Omega)\to B^p_{\mathrm{Cl}_n}(\Omega)$$

$$Q:L^p_{\mathrm{Cl}_n}(\Omega)\to D(W^{0,p,1}_{\mathrm{Cl}_n}(\Omega))$$

For $p=2$, these projections are ortho-projections. These projections have representation formula in terms of $\mathrm{tr}_{\sum}$, $\tilde{F}_{\sum}$ and $\tilde{T}_\Omega$. See References [2,12] for instance.

Next we search for special left-monogenic functions which are dense in the generalized Bergman p space $B^p_{\mathrm{Cl}_n}(\Omega)$. The functions will be defined from fundamental solutions of the Dirac operator and Laplacian. We choose dense points in an outer surface and construct functions from the fundamental solutions to have these points as their singularities. These functions then, are just shifted fundamental solutions or shifted kernels of the respective partial differential operators. Let us start with generalized concepts known in classical analysis.

Definition 4

Let X be a normed right-vector space over Cl_n. A system of points $\{x_m\colon m\in N\}\subset X$ is called $\mathrm{Cl}_{0,n}$ complete system in X, if the points approximate X finitely. I.e for each $\varepsilon>0$, for each $x\in X$, there exists $c_i\in\mathrm{Cl}_n$, $i=1,1,\ldots,n_0$ such that

$$\left\|x-\sum_{i=1}^{n_0}x_ic_i\right\|x<\varepsilon$$

Definition 5
A system of points $\{x_m\colon m\in N\}\subset X$ is called closed in X if every bounded Cl_n-valued right-linear functional F that vanishes on the points vanishes on the whole space X.

Lemma 5
The system of points $\{x_m\colon m\in N\}\subset X$ is closed if it is Cl_n-complete in X.

Thus, we have the following fundamental result over unbounded domains with C^2 boundaries.

Proposition 8
Let Ω and Ω_L be unbounded domains with C^2 boundaries such that $\Omega_L\supset \mathrm{cl}(\Omega)$. Let $\sum=\partial\Omega$ and $\sum_L=\partial\Omega_L$ be C^2 hypersurfaces. Let z be an arbitrary but fixed point in $R^n\backslash(\Omega)$, also $\{x_m\colon m\in N\}$ be a dense subset of $\sum_L$. Then the function system $Y=\{\Psi_m^z\colon m\in N\}$ forms a Cl_n-complete system in $B^p_{\mathrm{Cl}_n}(\Omega)$ for $1<p<\infty$.

Proof
From Lemma 5 above, it suffices to show that Y is closed. So we need that every $F\in B^p_{\mathrm{Cl}_n}(\Omega)^\star$ that vanishes on Y vanishes on $B^p_{\mathrm{Cl}_n}(\Omega)$. Here, $B^p_{\mathrm{Cl}_n}(\Omega)^\star$ is the space of all bounded Cl_n-valued continuous linear functionals on $B^p_{\mathrm{Cl}_n}(\Omega)$. So, let $F\in B^p_{\mathrm{Cl}_n}(\Omega)^\star$ such that $F=0$ on Y. But $F(\Psi^z(x,y))$ defines a monogenic f function on $R^n\backslash(\{z\}\cup\mathrm{cl}(\Omega))$. Moreover, $f(x)=0$ on a dense subset of $\partial\Omega_L$. A simple density argument shows that $f(x)=0$ on all of $\partial\Omega_L$. From Lusin's Theorem it follows that f is identically zero.

From the classical Hahn–Banach extension theorem, F has an extension say F^e to the whole space $L^p_{\mathrm{Cl}_n}(\Omega)$ with the same Clifford norm as F. From Riesz' linear functional representation theorem, the functional F^e can be given by: $F^e(\psi)=\int_\Omega \bar{g}\psi\,\mathrm{d}\Omega$ on the space $L^q_{\mathrm{Cl}_n}(\Omega)$ where, $g\in L^q_{\mathrm{Cl}_n}(\Omega)$ and $p^{-1}+q^{-1}=1$, with $p>1$ and $q>1$ and $F^e(\psi)=F(\psi)$ for $\psi\in B^p_{\mathrm{Cl}_n}(\Omega)$.

Let Ω_1 and Ω_2 be domains with C^2 boundaries and such that $\Omega\subset\Omega_1\subset\Omega_2$. It follows that $B^p_{\mathrm{Cl}_n}(\Omega_2)\subset B^p_{\mathrm{Cl}_n}(\Omega_1)\subset B^p_{\mathrm{Cl}_n}(\Omega)$. Furthermore by Proposition 6 for each $f\in B^p_{\mathrm{Cl}_n}(\Omega_2)$ and each $y\in\Omega$

$$f(y)=\frac{1}{\omega_n}\int_{\partial\Omega_1}\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

It follows that

$$F^e(f)=\int_\Omega g(y)f(y)\,\mathrm{d}y^n=\frac{1}{\omega_n}\int_\Omega g(x)\int_{\partial\Omega_1}\Psi^z(x,y)n(x)f(x)\,\mathrm{d}\sigma(x)$$

A simple application of Fubini's Theorem now reveals that $F^e(f)=0$ for each $f\in B^p_{\mathrm{Cl}_n}(\Omega_2)$. By taking inductive limits of the Banach spaces the result follows. □

Corollary
Under the hypothesis of the above theorem, the system $\{\Psi_m^z\colon m\in N\}$ is Cl_n-complete in $W^{p,k}_{\mathrm{Cl}_n}(\Omega)\cap B^p_{\mathrm{Cl}_n}(\Omega)$ for $k\in N\cup\{0\}$.

Lemma 6
Let $\|x\| > 2\max(\|y\|, \|z\|)$. Then:

$$\left\| \frac{1}{\|x-y\|^{n-2}} - \frac{1}{\|x-z\|^{n-2}} \right\| \leqslant \frac{(n-3)2^{n-1}\|y-z\|}{\|x\|^{n-1}}$$

Proof
Assume that $\|x\| > 2\max(\|y\|, \|z\|)$. Then

$$\begin{aligned}
&\|x\|^{n-1} \left\| \frac{1}{\|x-y\|^{n-2}} - \frac{1}{\|x-z\|^{n-2}} \right\| \\
&= \|x\| \left\| \left(\frac{\|x\|}{\|x-y\|} \right)^{n-2} - \left(\frac{\|x\|}{\|x-z\|} \right)^{n-2} \right\| \\
&= \|x\|^2 \left\| \frac{1}{\|x-y\|} - \frac{1}{\|x-z\|} \right\| \sum_{i=1}^{n-3} \left(\frac{\|x\|}{\|x-y\|} \right)^i \left(\frac{\|x\|}{\|x-z\|} \right)^{n-3-i} \\
&\leqslant (n-3)2^{n-1}\|y-z\|.
\end{aligned}$$

Then the inequality follows. □

Corollary
Let Ω be an unbounded domain in R^n, and let $z \in \mathrm{cl}(\Omega)^c$ be a fixed but arbitrary point. Let also $\Omega\varepsilon = \Omega \backslash B(y, \varepsilon)$ for $\varepsilon > 0$. Then:

$$\Psi_1^z(x, y) := \frac{1}{(2-n)\omega_n} \left(\frac{1}{\|x-y\|^{n-2}} - \frac{1}{\|x-z\|^{n-2}} \right) \in L^p_{\mathrm{Cl}_n}(\Omega_\varepsilon)$$

for $n/(n-1) < p < \infty$.

Proof
This follows from Lemma 6. □

Proposition 9
Let Ω, Ω_L be unbounded C^2 domains in R^n with $\Omega_L \supset \Omega$, and z be an arbitrary but fixed point in $R^n \backslash \mathrm{cl}(\Omega)$. Let $\{x_m\colon m \in N\} \subset \Omega_L$. For each $x \in \Omega$, define:

$$\Psi^z_{m,0}(x) := -\frac{1}{\omega_n} \left(\frac{x - x_m}{\|x-x_m\|^n} - \frac{x-z}{\|x-z\|^n} \right)$$

and

$$\Psi^z_{m,1}(x) := \frac{1}{(2-n)\omega_n} \left(\frac{1}{\|x-x_m\|^{n-2}} - \frac{1}{\|x-z\|^{n-2}} \right)$$

Then

$$\tilde{T}_\Omega(\Psi^z_{m,0})(x) = \Psi^z_{m,1}(x) + h$$

where, h is monogenic on Ω and $\Psi^z_{m,0} = \Psi^z_m$.

Proof

We have that $\Psi^z_{m,0}(x) = D_x(\Psi^z_{m,1}(x))$. Then taking the $\tilde{T}_\Omega$-transform of both sides of the above equation and using the Borel–Pompeiu formula, we get

$$\tilde{T}_\Omega \Psi^z_{m,0}(x) = \tilde{T}_\Omega D \Psi^z_{m,1}(x) = \Psi^z_{m,1}(x) - \tilde{F}_{\sum} \Psi^z_{m,1}(x)$$

Then taking $h = -\tilde{F}_{\sum} \Psi^z_{m,1} \in \ker D(\Omega)$, we get the result. □

Here is another fundamental result on completeness which is a generalization of a result of Gürlebeck and Sprößig [2].

Proposition 10

Let Ω and Ω_L be the domains in Proposition 9 with $\sum = \partial\Omega, \sum_L = \partial\Omega_L,$ and z as defined before. Let $\{x_m\colon m \in N\}$ be a dense subset of $\sum_L$. Then for $n/(n-1) < p < \infty$, the set

$$\{\Psi^z_{m,0}\colon m \in N\} \cup \{\Psi^z_{m,1}\colon m \in N\}$$

is Cl_n-complete in the space $\ker \triangle(\Omega) \cap L^p_{\mathrm{Cl}_n}(\Omega)$.

Proof

Let $f \in L^p_{\mathrm{Cl}_n}(\Omega) \cap \ker\triangle(\Omega)$ and let $\varepsilon > 0$ be given. Then for $k > 1$, there exists a sequence $\{f_j\} \subset W^{p,k+1}_{\mathrm{Cl}_n}(\Omega) \cap \ker\triangle(\Omega)$ such that $\lim_{j\to\infty} f_j = f$ in $L^p_{\mathrm{Cl}_n}(\Omega)$. Thus, there exists $n_0 \in N$ such that $\|f_j - f\|_{L^p} \leqslant \varepsilon/2$, for every $n \geqslant n_0$. But also from Lemma 4, there exist $g_1 \in \ker D(\Omega) \cap W^{p,k}_{\mathrm{Cl}_n}(\Omega)$, $g_2 \in \ker D(\Omega) \cap W^{p,k-1}_{\mathrm{Cl}_n}(\Omega)$ such that $f_{n_0} = g_1 + \tilde{T}_\Omega g_2$. From the Cl_n-completeness of $\{\Psi^z_m\colon m \in N\}$ over $\ker D(\Omega) \cap L^p_{\mathrm{Cl}_n}(\Omega)$, there exists $n_1 \in N$ and appropriate Clifford numbers $c_{i,n_1};\ i = 1, 2, \ldots, n_1$ with

$$\left\| g_2 - \sum_{i=1}^{n_1} \Psi^z_i c_{i,n_1} \right\|_{p,k} \leqslant \frac{\varepsilon}{4\|\tilde{T}_\Omega\|_{\mathrm{op}}}$$

where $\|\tilde{T}_\Omega\|_{\mathrm{op}}$ is the operator norm of $\tilde{T}_\Omega$. This implies:

$$\left\| \tilde{T}_\Omega\, g_2 - \tilde{T}_\Omega \left(\sum_{i=1}^{n_1} \Psi^z_i c_{i,n_1} \right) \right\|_p \leqslant \frac{\varepsilon}{4}$$

Also, since

$$\tilde{T}_\Omega \left(-\sum_{i=1}^{n_1} \Psi^z_{i,0} c_{i,n_1} \right) = \sum_{i=1}^{n_1} \Psi^z_{i,1}\, c_{i,n_1} + \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1}$$

where, $\psi_{i,\Omega} \in \ker D(\Omega)$, for $i = 1, 2, \ldots, n_1$, then denoting the expression $g_1 + \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1} \in \ker D(\Omega)$ by g_3, again from the completeness of $\{\Psi^z_m\colon m \in N\}$ in $B^p_{\mathrm{Cl}_n}(\Omega)$, there exists $n_2 \in N$ and Clifford numbers $b_{i,n_2},\ i = 1, 2, \ldots, n_2$ with

$$\left\| g_3 - \sum_{i=1}^{n_2} \Psi^z_{i,0}\, b_{i,n_2} \right\|_p \leqslant \frac{\varepsilon}{4}$$

Hence,

$$\begin{aligned}
&\left\| f_{n_0} - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{in_1} - \sum_{i=1}^{n_2} \Psi_i^z b_{i,n_2} \right\|_p \\
&= \left\| g_1 + \tilde{T}_\Omega g_2 - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} - \sum_{i=1}^{n_2} \Psi_{i,0}^z\, b_{i,n_2} \right\|_p \\
&= \left\| g_1 + \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1} - \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1} - \sum_{i=1}^{n_1} \Psi_{i,1}^z\, c_{i,n_1} - \sum_{i=1}^{n_2} \Psi_{i,0}^z b_{i,n_2} + \tilde{T}_\Omega g_2 \right\|_p \\
&= \left\| g_3 - \sum_{i=1}^{n_2} \Psi_i^z b_{i,n_2} + \tilde{T}_\Omega g_2 - \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1} - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} \right\|_p \\
&\leqslant \left\| g_3 - \sum_{i=1}^{n_2} \Psi_i^z b_{in,2} \right\|_p + \left\| \tilde{T}_\Omega g_2 - \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1} - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} \right\|_p \\
&= \left\| g_3 - \sum_{i=1}^{n_2} \Psi_i^z b_{i,n_2} \right\|_p + \left\| \tilde{T}_\Omega g_2 - \left(\sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} + \sum_{i=1}^{n_1} \psi_{i,\Omega} c_{i,n_1} \right) \right\|_p \\
&= \left\| g_3 - \sum_{i=1}^{n_2} \Psi_i^z b_{i,n_2} \right\|_p + \left\| \tilde{T}_\Omega g_2 - \tilde{T}_\Omega \left(\sum_{i=1}^{n_1} \Psi_{i,0}^z c_{i,n_1} \right) \right\|_p \\
&\leqslant \left\| g_3 - \sum_{i=1}^{2} \Psi_{i,0}^z b_{i,n_2} \right\| Lp + \|\tilde{T}_\Omega\| \left\| g_2 - \sum_{i=1}^{n_1} \Psi_{i,0}^z c_{i,n_1} \right\|_p \\
&\leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}
\end{aligned}$$

Thus,

$$\begin{aligned}
\left\| f - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} - \sum_{i=1}^{n_2} \Psi_{i,0}^z b_{i,n_2} \right\|_p &= \left\| f - f_{n_0} + f_{n_0} - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} - \sum_{i=1}^{n_2} \Psi_{i,0}^z\, b_{i,n_2} \right\|_p \\
&\leqslant \|f - f_{n_0}\|_p + \left\| f_{n_0} - \sum_{i=1}^{n_1} \Psi_{i,1}^z c_{i,n_1} - \sum_{i=1}^{n_2} \Psi_{i,0}^z\, b_{i,n_2} \right\|_p \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Therefore, the proposition is proved. □

Corollary
Let $n/(n-1) < p < \infty$ and z be an arbitrary but fixed point in $R^n \backslash (\Omega)$. Then, for each $k \in N$, the set, $\{\Psi_{m,0}^z; \Psi_{m,1}^z \colon m \in N\}$ is Cl_n-complete in the space $W_{\mathrm{Cl}_n}^{p,k}(\Omega) \cap \ker \triangle(\Omega)$.

4. SOME APPLICATIONS

The complete function systems we constructed above are now to be used to approximate solutions to partial differential equations of the respective order. Thus, we have the following results.

Proposition 11
Let $1 < p < \infty$, Ω be an unbounded C^2 domain in R^n and z be an arbitrary but fixed point in $R^n \setminus \mathrm{Cl}(\Omega)$. Let $u \in B^p_{\mathrm{Cl}_n}(\Omega)$. Then there exist Clifford numbers c_{i,n_1}; $i = 1, 2, \ldots, n_1$ such that for each $\varepsilon > 0$

$$\left\| u - \sum_{i=1}^{n_1} \Psi^z_{i,0} c_{i,n_1} \right\|_p < \varepsilon$$

on Ω.

Proposition 12
Let $n/(n-1) < p < \infty$, $g \in W^{p,k+2-1/p}_{\mathrm{Cl}_n}(\sum)$ and Ω be sufficiently smooth and unbounded domain in R^n and z be as above. Then for $u \in \ker(\triangle) \cap L^p_{\mathrm{Cl}_n}(\Omega)$ there exist Clifford numbers c_{i,n_1}; $i = 1, 2, \ldots, n_1$ and c_{i,n_2}; $i = 1, 2, \ldots, n_2$ such that for each $\varepsilon > 0$ we have

$$\left\| u - \sum_{i=1}^{n_1} \Psi^z_{i,0} c_{i,n_1} - \sum_{i=1} \Psi^z_{i,1} c_{i,n_2} \right\|_p < \varepsilon$$

on Ω.

REFERENCES

1. Gürlebeck K, Sprößig W. *Quaternionic and Clifford Calculus for Physicists and Engineers*. Wiley: Chichester, 1997.
2. Gürlebeck K, Sprößig W. *Quaternionic Analysis and Elliptic Boundary Value Problems*. Birkhauser: Basel, 1990.
3. Bernstein S. Operator calculus for elliptic boundary value problems in unbounded domains. *Zeitschrift für Analysis Und ihre Anwendungen* 1991; **10**(4):447–460.
4. Franks E, Ryan J. Bounded monogenic functions on unbounded domains. *Contemporary Mathematics* 1998; **212**:71–79.
5. Gurlebeck K, Kahler U, Ryan J, Sprossig W. Clifford analysis over unbounded domains. *Advances in Applied Mathematics* 1997; 216–239.
6. McIntosh A. Clifford algebras, Fourier theory, singular integral operators and partial differential equations on Lipschitz domains. In *Clifford Algebras in Analysis and Related Topics*, Ryan J (ed.). CRC Press: Boca Raton, 1996; 33–88.
7. Xu Z, Zhou C. On boundary value problems of Riemann–Hilbert type for monogenic functions in a half space of R^n for $n > 2$. *Complex Variables*: *Theory and its Application* 1993; **3–4**:181–193.
8. Zhou C. On boundary value problems of Neumann type for the Dirac operator in a half space R^n for $n > 2$. *Complex Variables*: *Theory and its Application* 1993, **1–2**:1–16.
9. Triebel H. *Interpolation Theory*, *Function Spaces*, *Differential Operators*. North-Holland Mathematical Library: Amsterdam, 1978.
10. Brackx F, Delanghe R, Sommen F. *Clifford Analysis*, Research Notes in Mathematics, vol. 76. Pitman: London, 1982.
11. Ryan J. Iterated Dirac operators and conformal transformations in $\mathbb{R}^n$. In *Proceedings of the 18th International Conference on Differential Geometric Methods in Theoretical Physics*. World Scientific: Singapore, 1987; 390–399.

12. Kähler, U. Elliptic boundary value problems in bounded and unbounded domains. In *Dirac Operators in Analysis*, Pitman Research Notes in Mathematics Series, vol. 394. Longman Science Technology: Harlow, 1998; 122–140.
13. Ryan J. Basic Clifford analysis. *Cubo Matematica Educacional* 2000; **2**:226–256.
14. Coifman R, McIntosh A, Meyer Y. L'integrale de Cauchy definit un operateur borne sur L' pour les courbes Lipschitziennes. *Annals of Mathematics* 1982; **116**:361–387.
15. Mitrea M. *Singular Integrals*, *Hardy Spaces*, *and Clifford Wavelets*. Lecture Notes in Mathematics, vol. 1575. Springer-Verlag: Heidelberg, 1994.

