MAT 210 PREVIEW MID TERM EXAM SPRING 2010:

Part I. Multiple Choice. The correct answer has a *check mark* next to it.

1. Evaluate the limit: $\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$

   a. 4 ✓ b. 2 c. $\nexists$ d. ∞

2. For the function $f(x) = 3x + 7$ what is the derivative $f'(x)$ of f.

   a. $f'(x) = 3$ ✓ b. $f'(x) = 0$ c. $f'(x) = 2x$ d. None

3. The derivative of $f(x) = \frac{2}{x}$ is :

   a. 2 b. $\frac{-2}{x^2}$ ✓ c. x d. 0

   For problems # 4−7 only: For the function $f(x) = \begin{cases} x^2 + 2x & \text{, if } x \geq 3 \\ 3x + \alpha & \text{, if } x < 3 \end{cases}$

4. What is : $\lim\limits_{x \to 3^+} f(x)$

   a. 15✓ b. 12 c. $9 + \alpha$ d. α

5. What is : $\lim\limits_{x \to 3^-} f(x)$

   a. 15 b. 12 c. $9 + \alpha$✓ d. α

6. What is : $\lim\limits_{x \to 3} f(x)$

   a. 15 b. 12 c. $9 + \alpha$ d. $\nexists$✓

7. The value of α so that f is continuous at $x = 3$

   a. $\alpha = 6$✓ b. $\alpha = 7$ c. $\alpha \neq 6$ d. $\alpha = 10$

   ★ For problems # $8 - 11$ only. From the top of a building of height $60ft$, a ball is thrown vertically upwards with an initial velocity of $36ft/\sec$.

   Then the height the ball reaches t seconds after it is released is given by $h(t) = -16t^2 + 36t + 60\ ft$ .Then

8. the height of the ball when $t = 2$ seconds is:

   Solution: $h(2) = -16(2)^2 + 36(2) + 60 = 68ft$

9. The time at which the ball reaches the highest point (same as the time at which the upward velocity of the ball is zero) is:

   Answer: This is the time where the velocity is zero. But the velocity at any time t is :

$v(t)=\frac{dh(t)}{dt}=\frac{d}{dt}(-16t^2+36t+60)=-32t+36.$

Therefore setting $v(t)=0 \Longrightarrow -32t+36=0$

$\Longrightarrow 32t=36$

$\Longrightarrow t=\frac{36}{32}\sec=\frac{9}{8}\sec$

Thus the ball stops moving upwards when: $t=\frac{36}{32}\sec$ when the velocity is zero.

10. The time where the ball touches the ground . Here set $h(t)=0$ and solve for t.

    Answer: The time where the ball touches the ground is the time t such that $h(t)=0$.

    Then solving the equation : $h(t)=-16t^2+36t+60=0$

    Using the quadratic formula we solve for t and get: $t=\frac{9\pm2\sqrt{69}}{8}$ with two distinct values :

    $t_1=\frac{9+2\sqrt{69}}{8}=\frac{1}{4}\sqrt{69}+\frac{9}{8}=3.2017\sec$ and $t_2=\frac{9-2\sqrt{69}}{8}=-0.95166\sec$

    But $t_2=\frac{9-2\sqrt{69}}{8}=-0.95166\sec$ is negative time and therefore we choose the first value of t.

    Hence it is when $t=3.2017\sec$ that the ball touches the ground.

11. The velocity of the ball at the instant it hits the ground is :.

    Answer: The velocity at which the ball touches the ground is the velocity v when $t=3.2017\sec$

    which is : $v(3.2017)=v(t)_{|t=3.2017\sec}ft/\sec$

    $=(-32t+36)_{|t=3.2017\sec}ft/\sec$

    $=-32(3.2017)+36ft/\sec$

    $=-66.454ft/\sec$

PART II. WORK OUT (Show all the necessary steps)

12. Show that the function given by :

    $.f(x)=\begin{cases} x^2+4, & x>0 \\ -x^2+\alpha, & x\le 0 \end{cases}$

    is continuous at $x=0$ when $\alpha=4$

    Solution: We need to check conditions of continuity :

    $$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}(x^2+4)=4 \text{ and } \lim_{x\to0^-}f(x)=\lim_{x\to0^-}(-x^2+\alpha)=\alpha$$

    and the value of the function f at 0 : $f(0)=\alpha$.

Therefore for the function f to be continuous at $x = 0$, the three numbers should be the same;

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$ and from these equalities we get : $\alpha = 4$

13. Use the definition of the derivative to find $f'(x)$ for : $f(x) = 3x^2 + x - 1$

    That is : $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

    Solution: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

    $$= \lim_{h\to 0} \frac{3(x+h)^2+(x+h)-1-\left(3x^2+x-1\right)}{h}$$

    $$= \lim_{h\to 0} \frac{6xh+3h^2+h}{h} = \lim_{h\to 0} \frac{h(6x+3h+1)}{h}$$

    $$= \lim_{h\to 0} (6x + 3h + 1) = 6x + 1$$

    $\therefore f'(x) = 6x + 1$

14. For the function in problem #13, find the equations of the tangent and normal lines to the graph at the point $(1, 3)$

    First find slope $m = f'(1) = f'(x)_{|x=1}$. Next use slope point form : $y - y_1 = m(x - x_1)$ to determine the equation.

    Solution: Thus the slope of the tangent line is : $m = f'(1) = f'(x)_{|x=1} = (6x + 1)_{|x=1} = 7$

    and the line passes through $(1, 3)$. Then using the slope-point form : $y - y_1 = m(x - x_1)$:

    which is : $y - 3 = 7(x - 1) \Leftrightarrow y = 7x - 4$

    $\therefore \boxed{y = 7x - 4}$ equation of the tangent line.

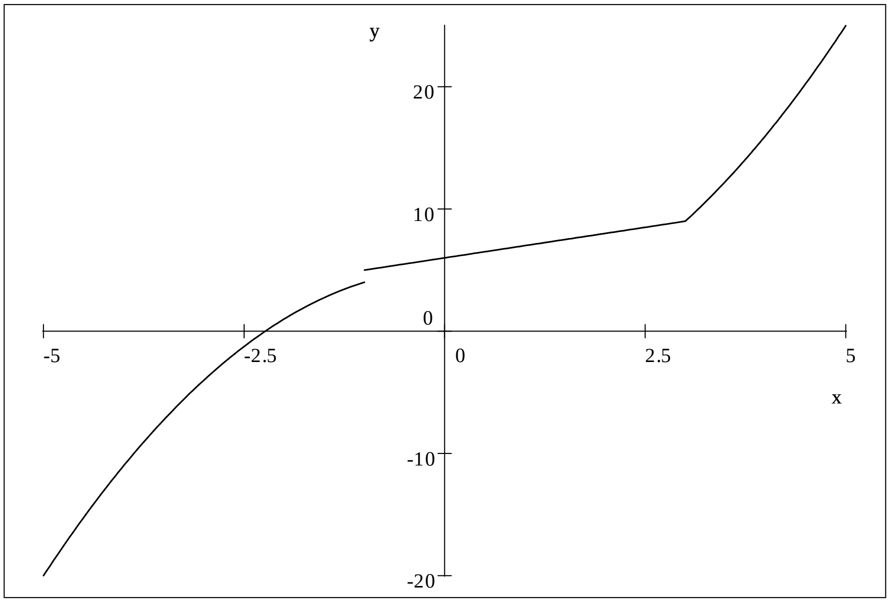
    The normal line is the line that passes through the same point of contact but perpendicular to the tangent line.

    Therefore it has slope the negative reciprocal of the slope of the tangent line which is : $\frac{-1}{7}$.

    Hence the equation of the normal line will be: $y - 3 = \frac{-1}{7}(x - 1) = \frac{-1}{7}x + \frac{1}{7}$.

    Solving for y we have : $\boxed{y = \frac{-1}{7}x + \frac{22}{7}}$ equation of thye normal line.

15. Identify the points where the function whose graph given below is not differentiable. Provide your reasons:



Answer: At points where the graph is discontinuous and where the graph has a corner point : $x = -1$ and $x = 3$

16. Identify the points where the function defined by : $f(x) = \begin{cases} x^2 & \text{if} \quad x \geq 3 \\ x+6 & \text{if} \quad -1 < x < 3 \\ -x^2+5 & \text{if} \quad x < -1 \end{cases}$

is not differentiable?

Solution: It is clear that when $x \neq -1, 3$ the function is differentiable, because on each sub interval :

on $(-\infty, -1)$, the function $f(x) = -x^2 + 5$ which is differentiable.

on $(-1, 3)$ the function $f(x) = x + 6$ which is again differentiable there and finally

on $(3, \infty)$ the function is $f(x) = -x^2 + 5$ which is differentiable as well.

But then at $x = -1$, the function is discontinuous and hence not differentiable and at $x = 3$ we check the one sided derivatives of f :

Right hand derivative of f at $x = 3$: $f'(3^+) = \lim_{x \to 3^+} \frac{f(x)-f(3)}{x-3}$

$$= \lim_{x \to 3^+} \frac{x^2-9}{x-3} = \lim_{x \to 3^+} \frac{(x-3)(x+3)}{x-3}$$

$$= \lim_{x \to 3^+} (x+3) = 6.$$

Left hand derivative of f at $x=3: f'\left(3^{-}\right)=\lim_{x\to 3^{-}}\frac{f(x)-f(3)}{x-3}=\lim_{x\to 3^{-}}\frac{x+6-9}{x-3}$

$$=\lim_{x\to 3^{-}}\frac{x-3}{x-3}=1.$$

We therefore see : $f'(3^{+})\neq f'(3^{-})$

$\therefore$ f is not differentiable at $x=3$

17. A train is moving in a straight line whose distance s at time t is given by $s(t)=4t^2+10$ miles in t hrs. Then compute

a. the average velocity of the train on $[3,3+h]$: $\frac{s(3+h)-s(3)}{h}$

Solution: the average velocity of the train on $[3,3+h]$ is :

$$\frac{s(3+h)-s(3)}{h}=\frac{4(3+h)^2+10-\left(4(3)^2+10\right)}{h}$$
$$=\frac{36+24h+4h^2+10-46}{h}=\frac{24h+4h^2}{h}$$
$$=\frac{h(24+4h)}{h}=24+4h$$

b. the instantaneous velocity (or instantaneous rate of change) of the train exactly at $t=3$ hrs:

$$v\left(3\right)=s'(3):=\lim_{h\to 0}\frac{s(3+h)-s(3)}{h}$$

Solution: $v\left(3\right)=s'(3):=\lim_{h\to 0}\frac{s(3+h)-s(3)}{h}ft/\sec=\lim_{h\to 0}\left(24+4h\right)ft/\sec=$ $24ft/\sec$

18. Show that the function : $f\left(x\right)=\mid x\mid$ is not differentiable at $x=0$, by showing :

the right hand derivative of f at 0: $f'\left(0^{+}\right)$ and $f'(0^{-})$: left hand derivative of f at 0 are unequal.

Solution: Here you have to use the fact that : $\mid x\mid=\begin{cases} x & \text{if } x\geq 0\\ -x & \text{if } x<0\end{cases}$

Therefore, the right hand derivative of f at $0: f'\left(0^{+}\right)=\lim_{x\to 0^{+}}\frac{f(x)-f(0)}{x-0}=$

$$\lim_{x\to 0^{+}}\frac{|x|-|0|}{x}\underset{|x|=x}{\underset{\uparrow}{=}}\lim_{x\to 0^{+}}\left(\frac{x}{x}\right)=1$$

and the left hand derivative of f at $0: f'\left(0^{-}\right)=\lim_{x\to 0^{-}}\frac{f(x)-f(0)}{x-0}=$

$$\lim_{x\to 0^{-}}\frac{|x|-|0|}{x}\underset{|x|=-x}{\underset{\uparrow}{=}}\lim_{x\to 0^{-}}\left(\frac{-x}{x}\right)=-1$$

Hence the right hand derivative and the left hand derivative of f at 0 are different.

$\therefore$ $f'(0)\nexists$. That is $f'(0)$ does not exist.

19. Show that if f is a function which is differentiable at $x = 3$, then it is continuous at that point.

Hint: use: $f(x) - f(3) = \frac{f(x)-f(3)}{x-3}(x-3)$ and compute the $\lim\limits_{x\to 3}$ of both sides of the equality.

Then use the fact : $f'(3) = \lim\limits_{x\to 3} \frac{f(x)-f(3)}{x-3}$.

Solution: $f(x) - f(3) = \frac{f(x)-f(3)}{x-3}(x-3)$ for $x \neq 3$.

Then taking the $\lim\limits_{x\to 3}$ of both sides of the equation we have ;

$$\lim_{x\to 3}(\ f(x) - f(3)) = \lim_{x\to 3}\left(\frac{f(x)-f(3)}{x-3}(x-3)\right)$$

$$= \lim_{x\to 3}\frac{f(x)-f(3)}{x-3}.\lim_{x\to 3}(x-3) = f'(3).0 = 0$$

That is : $\lim\limits_{x\to 3}(\ f(x) - f(3)) = 0$

$$\Rightarrow \lim_{x\to 3} f(x) = \lim_{x\to 3} f(3) = f(3)$$

$\therefore f$ is continuous at $x = 3$

20. Use rules of differentiations to find the derivative of the following function:

$$f(x) = x^4 + 3x^2 - 7\sqrt{x}$$

Solution: $f'(x) = \frac{d}{dx}\left(x^4 + 3x^2 - 7\sqrt{x}\ \right) = \frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(3x^2\ \right) - \frac{d}{dx}\left(7\sqrt{x}\ \right)$

$$= 4x^3 + 6x - 7\frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= 4x^3 + 6x - 7\left(-\frac{1}{2}\right)x^{1-\frac{1}{2}}$$

$$= 4x^3 + 6x + \frac{7}{2}x^{-\frac{1}{2}}$$

21. Verify the Intermediate value theorem for the function : $f(x) = x^2 + 3x - 1$ on the interval $[-1, 0]$ for $M = -2$.

Solution: The function $f(x) = x^2 + 3x - 1$ is a polynomial function and it is continuous on $\mathbb{R}$ and therefore it is continuous on the given closed interval $[-1, 0]$.

Besides $f(-1) = -3$ and $f(0) = -1$ and the value $M = -2$ is an intermediate value of f between $-3 = f(-1)$ and $-1 = f(0)$.

Then by the intermediate value theorem there exists a number c in the interval $[-1, 0]$ such that $f(c) = M = -2$.

Then solving $f(c) = c^2 + 3c - 1 = -2$ for c , we have :

$c^2 + 3c + 1 = 0 \Rightarrow c = -\frac{1}{2}\sqrt{5} - \frac{3}{2} = -2.618$ or $c = \frac{1}{2}\sqrt{5} - \frac{3}{2} = -0.38197$.

The first vcalue of $c = -2.618$ is outside of the given interval but the second value of $c = -0.38197$ is in the given interval.

Therefore the required number is $c = \frac{1}{2}\sqrt{5} - \frac{3}{2}$

22. Show that the function : $f(x) = x^3 - 3x + 1$ has a zero on the interval $[0, 1]$.

Solution: The function $f(x) = x^3 - 3x + 1$ is continuous on $[0, 1]$ and $f(0) = 1$ and $f(1) = -1$.

Then by the intermediate value theorem for $M = 0$, there exists a number $c \in (0, 1)$ such that $f(c) = 0$.

$\therefore$ The value $x = c$ is the zero of the function on the given interval.

23. Differentiate the function: $f(x) = x^2 (x+1)^3$

Solution: $f'(x) = \frac{d}{dx}\left(x^2 (x+1)^3\right) = \left(\frac{d}{dx}(x^2)\right)(x+1)^3 + x^2 \frac{d}{dx}(x+1)^3$

$$= 2x (x+1)^3 + x^2 \left(3 (x+1)^2 .(1+0)\right)$$

$$= 2x(x+1)^3 + 3x^2 (x+1)^2$$

$$= x (x+1)^2 (2(x+1) + 3x)$$

$$= x (5x+2)(x+1)^2$$

24. Differentiate the function : $f(x) = \frac{x}{x^2+1}$

Solution: Use quotient rule of differentiation:

$$f'(x) = \frac{d}{dx}\left(\frac{x}{x^2+1}\right) = \frac{\left(\frac{d}{dx}(x)\right)\left(x^2+1\right) - x\frac{d}{dx}\left(x^2+1\right)}{(x^2+1)^2}$$

$$= \frac{1\left(x^2+1\right) - x(2x)}{(x^2+1)^2}$$

$$= \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

25. Differentiate: $h(x) = (f \circ g)(x)$ where $f(x) = x^4$ and $g(x) = 5x^2 + 3x + 1$

Solution: We use chain rule to differentiate compositions of functions. Thus,

$h'(x) = ((f \circ g)(x))' = (f(g(x)))' = f'(g(x)).g'(x)$

But $f'(x) = 4x^3$ and $g'(x) = \left(5x^2 + 3x + 1\right)' = 10x + 3$.

Therefore, $h'(x) = ((f \circ g)(x))' = (f(g(x)))'$

$$= f'(g(x)).g'(x) = 4 (g(x))^3 . (10x+3)$$

$$= 4 \left(5x^2 + 3x + 1\right)^3 (10x + 3)$$

$$= (40x + 12) \left(5x^2 + 3x + 1\right)^3$$