An other Look at Higher Order Derivatives:

Theorem 1 *(Dejenie A. Lakew) Let* Ω *be a non-empty open subset of* $\mathbb{R}$ *and* $f \in C^{\infty}(\Omega, \mathbb{R})$*. Then for* $x_0 \in \Omega$*,*

$$f^{(n)}(x_0) = \lim_{h_n \to 0} \left(\lim_{h_{(n-1)} \to 0} \left(\lim_{,h_{(n-2)} \to 0} \left(\cdots \left(\lim_{h_1 \to 0} \left(\frac{f\left(x_0 + \sum_{i=1}^{n} h_i\right) - f\left(x_0 + \sum_{i=2}^{n} h_i\right) - \ldots - f(x_0 + h_1) + f}{\prod_{i=1}^{n} h_i} \right.\right.\right.\right.\right.$$

or thaking all the $h_i's$ *equal to* h*, we have*

$$f^{(n)}(x_0) = \lim_{h \to 0} \left(\frac{f(x_0 + nh) - f(x_0 + (n-1)h - \ldots - f(x_0 + h) + f(x_0))}{h^n} \right)$$

In particular, the second derivative of f *at* x_0 *therefore is given by :*

$$f^{(2)}(x_0) = \lim_{h_2 \to 0} \left(\lim_{h_1 \to 0} \left(\frac{f(x_0 + h_2 + h_1) - f(x_0 + h_2) - f(x_0 + h_1) + f(x_0)}{h_2 h_1} \right) \right)$$

or taking all the $h_i's$ to be h, we have

$$f''(x_0) = \lim_{h \to 0} \left(\frac{f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)}{h^2} \right)$$

Example 2 *For the function* $f(x) = x^3$*, we know that the second derivative is :* $f''(x) = 6x$

Then verifying the above definition for the second order derivative we have:

$$f''(x) = \lim_{h \to 0} \left(\frac{(x + 2h)^3 - 2(x + h)^2 + x^3}{h^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{x^3 + 6x^2h + 12xh^2 + 8h^3 - 2x^3 - 6x^2h - 6xh^2 - 2h^3 + x^3}{h^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{6xh^2 + 6h^3}{h^2} \right)$$

$$= \lim_{h \to 0} \left(\frac{h^2 \left(6x + 6h\right)}{h^2} \right)$$

$$= \lim_{h \to 0} \left(6x + 6h\right) = 6x$$