MAT 210 Preview Problems & Solutions for Final Exam Fall 2009

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1. Evaluate:

   a. $\lim\limits_{x\to 5}\frac{x^2-25}{x-5}$.

   Solution: $\lim\limits_{x\to 5}\frac{x^2-25}{x-5}=\lim\limits_{x\to 5}\frac{(x-5)(x+5)}{x-5}=\lim\limits_{x\to 5}(x+5)=10$

   b. $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$ for $f(x)=\frac{4}{x}$

   Solution:
   $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{\frac{4}{x+h}-\frac{4}{x}}{h} \\ &= \lim_{h\to 0}\frac{\frac{4x-4(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\left(\frac{-4h}{x(x+h)}\right) \\ &= \lim_{h\to 0}\left(\frac{-4}{x(x+h)}\right)=\frac{-4}{x^2} \end{aligned}$$

2. Show that the function : $f(x)=\begin{cases} x^2+3, x>0 \\ -x^2+\alpha, x\le 0 \end{cases}$,

   is continuous at $x=0$ when $\alpha=3$.

   Solution: Evaluating the one sided limits of f at $x=0$,

   RHL: $f(0^+)=\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}(x^2+3)=3$

   LHL: $f(0^-)=\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^+}(-x^2+\alpha)=\alpha$.

   Then the function f is continuous at $x=0$ when $RHL=LHL$ at $x=0$ which means

   $\alpha=3$. Therefore when $\alpha\neq 3$, the function is discontinuous at $x=0$.

3. Using the definition, find the derivative of the function : $f(x)=3x^2+x+2$.

   Solution:
   $$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{3(x+h)^2+(x+h)+2-3x^2-x-2}{h} \\ &= \lim_{h\to 0}\frac{6xh+3h^2+h}{h}=\lim_{h\to 0}\frac{h(6x+3h+1)}{h} \\ &= \lim_{h\to 0}(6x+3h+1)=6x+1 \end{aligned}$$

4. For the function given in problem #2 above, find the equation of the tangent line to the graph of f at $(1,6)$.

   Solution : $f'(x)=6x+1$ and therefore the slope m of the tangent line to the graph of the function at $(1,6)$ is given by

$m = f'(1) = 7$. Then using slope point form of equation of a straight line we determine the equation of the tangent line to be:

$: y - 6 = 7(x - 1)$. That is $l_{\tan} : y = 7x - 1$ is the required line.

5. The distance or position of a moving vehicle is given by the function $s(t) = 3t^2 - t + 10$ where t is in hrs and s is in miles.

   Compute:

   a. the average rate of change of s on $[2, 2 + h]$ is :

$$\frac{s(2+h) - s(2)}{h} = \frac{3(2+h)^2 - (2+h) + 10 - 20}{h} = \frac{11h + 3h^2}{h} = 11{+}3h$$

   b. the instantaneuous velocity of the vehicle at $t = 2$ hrs denoted by $v(2)$ is given by :

$$v(2) = s'(2) = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} \frac{11h + 3h^2}{h} = \lim_{h \to 0} (11 + 3h) = 11\text{miles/hr.}$$

   c. the velocity of the car at any time t and the time intervla where velocity remains positive.

   Solution: The velocity v at any time t : $v(t)$ is given by:

$$v(t) = \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \to 0} \frac{3(t+h)^2 - (t+h) + 10 - 3t^2 + t - 10}{h}$$

$$= \lim_{h \to 0} \frac{6th + 3h^2 - h}{h} = \lim_{h \to 0} (6t + 3h - 1) = 6t - 1$$

   Then the velocity remains positive : $v(t) > 0$ when $s'(t) = v(t) = 6t - 1 > 0 \Rightarrow t > \frac{1}{6}\ hrs = 10$ minutes.

6. Show that the piece-wise defined function : $f(x) = \begin{cases} x^2 + 2, x \geq 1 \\ 4x - 1, x < 1 \end{cases}$ is not differentiable at $x = 1$.

   Solution: We will do this by showing the two sided derivatives of the function at $x = 1$ are not equal. Indeed,

   RHL (Right Hand Limit):

$$f'\left(1^+\right) := \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{x^2 + 2 - 3}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1^+} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1^+} (x + 1) = 2$$

LHL (Left Hand Limit):

$$f'\left(1^{-}\right) := \lim_{x\to 1^{-}} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^{-}} \frac{4x-1-3}{x-1}$$

$$= \lim_{x\to 1-} \frac{4x-4}{x-1} = \lim_{x\to 1^{-}} \frac{4\left(x-1\right)}{x-1} = 4.$$

Thus RHL : $f'\left(1^{+}\right) = 2 \neq 4 = f'\left(1^{-}\right)$:LHL

$\therefore$ f is not differentiable at $x=1$.

7. Differentiate the following functions using some rules of differentiations:

a. $f(x) = (x+1)^3\left(x^2 +\right)$:

Solution: using product and chain rules :

$$f'\left(x\right) = \left((x+1)^3\right)'\left(x^2 +\right) + (x+1)^3\left(x^2 +\right)'$$

$$= 3\left(x+1\right)^2\left(x^2+1\right) + 2x\left(x+1\right)^3 = \left(x+1\right)^2\left(5x^2+2x+3\right)$$

b. $g(x) = e^{x^2+2x}$.

Solution: From chain rule and differentiating exponential functions:

$$g'\left(x\right) = \left(e^{x^2+2x}\right)' = e^{x^2+2x}\left(x^2+2x\right)' = \left(2x+2\right)e^{x^2+2x}$$

c. $h(x) = \frac{x}{x^3+1}$.

Solution: Using quotient rule:

$$h'(x) = \left(\frac{x}{x^3+1}\right)' = \frac{1\left(x^3+1\right) - x\left(3x^2\right)}{\left(x^3+1\right)^2} = \frac{-2x^3+1}{\left(x^3+1\right)^2}.$$

d. $k\left(x\right) = \ln\left(x^2+12\right)$.

Solution: Using chain rule and differentiation the natural logarithm, we have:

$$k'\left(x\right) = \frac{1}{x^2+12}.2x = \frac{2x}{x^2+12}$$

8. For the function : $f(x) = x^3 - 4x$

a Intervals of monotonicity(intervlas where $f \nearrow$ and where $f \searrow$) :

Solution: We use the first derivative for this: $f'(x) = 3x^2 - 4$.

Then

$$f'(x) > 0 \Rightarrow 3x^2-4 = 3\left(x^2 - \frac{4}{3}\right) = 3\left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) > 0 \Rightarrow x \in (-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$$

$\therefore$ f is increasing ($f \nearrow$) on : $(-\infty, -\frac{2}{\sqrt{3}}) \cup (\frac{2}{\sqrt{3}}, \infty)$

And

$$f'(x) < 0 \Rightarrow 3x^2-4 = 3\left(x^2 - \frac{4}{3}\right) = 3\left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) < 0 \Rightarrow x \in \left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$$

$\therefore f$ is decreasing ($f \searrow$) on $\left(\frac{-2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

b. Local extrema using the first derivative:

Solution: Using the sign of the first derivative, we see that f' changes sign from $(+)$ to $(-)$ at $x = -\frac{2}{\sqrt{3}}$ and from $(-)$ to $(+)$ at $x = \frac{2}{\sqrt{3}}$ and the numbers $x = \pm\frac{2}{\sqrt{3}}$ are in the domain of f

$$\therefore f\left(-\frac{2}{\sqrt{3}}\right) = \left(-\frac{2}{\sqrt{3}}\right)^3 - 4\left(-\frac{2}{\sqrt{3}}\right) = -\frac{8}{3\sqrt{3}} + \frac{8}{\sqrt{3}} = \frac{16}{2\sqrt{3}}$$

is the local maximum and

$$f\left(\frac{2}{\sqrt{3}}\right) = \left(\frac{2}{\sqrt{3}}\right)^3 - 4\left(\frac{2}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}} - \frac{8}{\sqrt{3}} = \frac{-16}{3\sqrt{3}}$$

is the local minimum of f

c. Intervals of concavity:

Solution: We use the second derivative here:

$$f'(x) = 3x^2 - 4 \Rightarrow f''(x) = 6x$$

Then

$$f''(x) > 0 \Rightarrow 6x > 0 \Rightarrow x > 0.$$

Thus f is concave up (f is $\cup$) on $(0, \infty)$.

Similarly,

$$f''(x) < 0 \Rightarrow 6x < 0 \Rightarrow x < 0$$

Thus f is concave down (f is $\cap$) on $(-\infty, 0)$

d. Inflection point: The function changes concavity from being concave up to concave down at point $(0, f(0)) = (0, 0)$ and therefore $(0, 0)$ is the inflection point to the graph of the function.

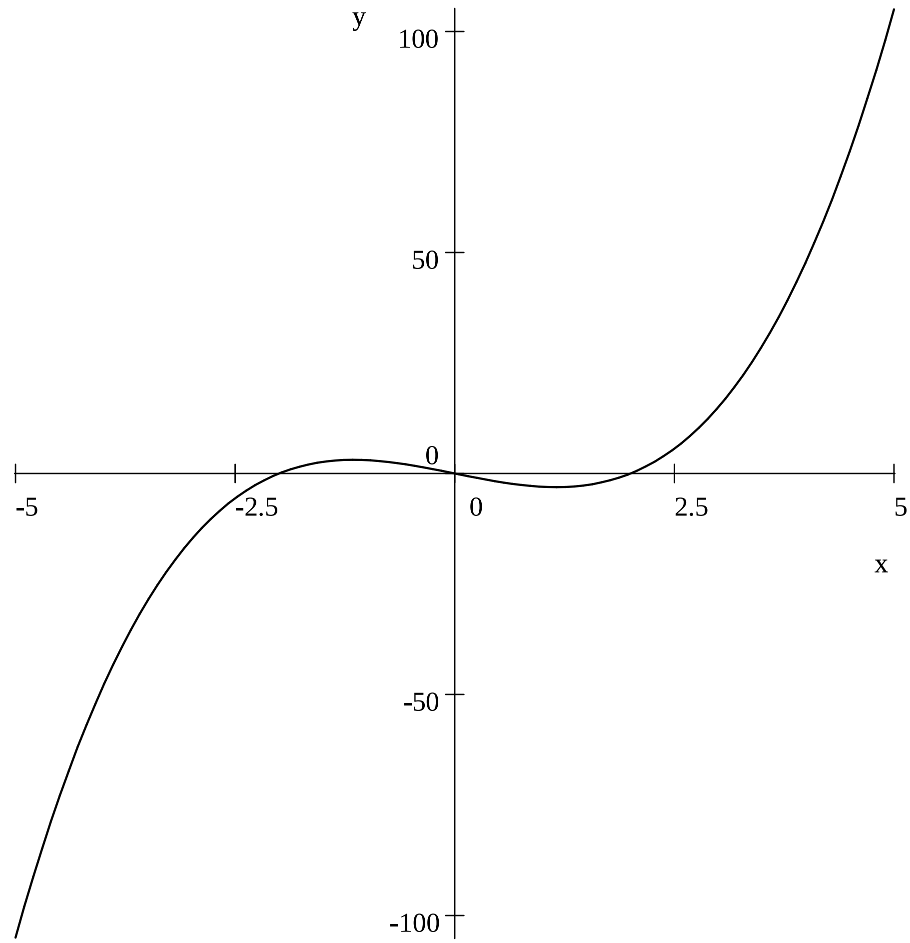
e. Axes intercepts: X-intercepts: $x's$:

$$f(x) = 0 \Rightarrow x^3-4x = 0 \Rightarrow x^3-4x = x\left(x^2 - 4\right) = x\,(x-2)\,(x+2) = 0 \Rightarrow x = 0, -2, 2$$

are the x-intercepts.

Y-intercept: $y = f(0) = 0$

f. Sketch of the graph of f :



Graph of $f(x) = x^3 - 4x$ on $[-5, 5]$

9. Evaluate the indefinite integrals:

a. $\int \left(x^3 - 5\sqrt[7]{x} + 81\right) dx$

Solution: $\int \left(x^3 - 5\sqrt[7]{x} + 81\right) dx = \frac{x^{3+1}}{3+1} - 5\frac{x^{\frac{1}{7}+1}}{\frac{1}{7}+1} + 81x + C$

$= \frac{x^4}{4} - 5\left(\frac{7}{8}\right)x^{\frac{8}{7}} + 81x + C = \frac{x^4}{4} - \frac{35}{8}x^{\frac{8}{7}} + 81x + C$

b. $\int \left(4x^3 + 2x\right)\left(x^4 + x^2\right)^{25} dx$: Use integration by substitution :

Solution: Let us make the substitution:

$$u = x^4 + x^2 \Rightarrow du = \left(x^4 + x^2\right)' dx = \left(4x^3 + 2x\right) dx$$

Then

$$\begin{aligned} \int \left(4x^3 + 2x\right)\left(x^4 + x^2\right)^{25} dx &= \int \Big(\underbrace{x^4 + x^2}_{\substack{\Vert \\ u}}\Big)^{25} \underbrace{\left(4x^3 + 2x\right) dx}_{\substack{\Vert \\ du}} \\ &= \int u^{25} du = \frac{u^{26}}{26} + C = \frac{\left(x^4 + x^2\right)^{26}}{26} + C \end{aligned}$$

10. Evaluate the definite integrals:

a. $\int\limits_{-1}^{1} \left(x^3 + 2x + 1\right) dx$

Solution: $\int\limits_{-1}^{1} \left(x^3 + 2x + 1\right) dx = \left(\frac{x^4}{4} + x^2 + x\right) |_{-1}^{1} = \left(\frac{1}{4} + 1 + 1\right) - \left(\frac{1}{4} + 1 - 1\right) =$ 2

b. $\int\limits_{0}^{1} \left(3x^2 + 2x\right)\left(x^3 + x^2\right)^5 dx.$

Solution: Using the substitution:

$$u = x^3 + x^2 \Rightarrow du = \left(x^3 + x^2\right)' dx = \left(3x^2 + 2x\right) dx$$

and making change of limits of integrations: when x is changing from $x = 0$ to $x = 1$ the new variable $u = x^3 + x^2$ is changing from 0 to 2.

$$\begin{aligned} \therefore \quad & \int\limits_{0}^{1} \left(3x^2 + 2x\right)\left(x^3 + x^2\right)^5 dx = \int\limits_{0}^{1} \left(x^3 + x^2\right)^5 \left(3x^2 + 2x\right) dx \\ = \quad & \int\limits_{0}^{2} u^5 du = \frac{u^6}{6} |_0^2 = \frac{2^6}{6} - \frac{0^6}{6} = \frac{2^6}{6} = \frac{32}{3} \end{aligned}$$

11. Compute the area of the region bounded by graphs of functions:

a. region bounded by : graph of $f(x) = 4 - x^2$, the x-axis, by $x = -4, x = 4$

Solution: Here the function $f(x) = 4 - x^2 \geq 0$ on the interval $[-2, 2]$. Thus the area of the region bounded by graph of f, and the x-axis on $[-2, 2]$ is given by the definite integral:

$$Area = A = \int_{-2}^{2} \left(4 - x^2\right) dx = \left(4x - \frac{x^3}{3}\right) |_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3} = \frac{32}{3}$$

b. region bounded by graphs of $f(x) = 4 - x^2$ and $g(x) = x^2 - 4$:

First the intersection numbers for the two functions are $x's$ such that $f(x) = g(x)$.

That is $4 - x^2 = x^2 - 4 \Rightarrow x = \pm 2$ and on the interval $[-2, 2]$ the function $f(x) \geq g(x)$.

$\therefore$ The area of the region described above is given by the definite integral:

$$Area = A = \int_{-2}^{2} \left(f_{\text{upper}} - f_{\text{lower}}\right) dx = \int_{-2}^{2} \left(f(x) - g(x)\right) dx$$

$$= \int_{-2}^{2} \left(4 - x^2 - \left(x^2 - 4\right)\right) dx = \int_{-2}^{2} \left(8 - 2x^2\right) dx$$

$$= \left(8x - \frac{2x^3}{3}\right) |_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = 32 - \frac{32}{3} = \frac{64}{3}$$

12. (On marginal functions): Demand equation for a Mathematics text book in a particular semester is :

$$p = -0.06x + 300 \, (0 \leq x \leq 5000)$$

Find:

a. Revenue function : $R(x) = xp(x) = x\left(-0.06x + 300\right) = -0.06x^2 + 300x$

b. Marginal revenue function:

Solution: The marginal revenue is the derivative of the revenue function. Thus $R'(x) = -0.12x + 300$

c. The actual revenuue to be gained from the sell of the 1001^{st} text book using the marginal revenue function.

Solution: You have learnt in class that :$R'(1000)$ is the revenue that is gained from the sell of the 1001^{st} text book.

Thus $R'(1000) = -0.12\,(1000) + 300 = -120 + 300 = 180$. That is \$180.00 is the actual revenue that will be gained from the sell of the 1001^{st} product.

d. The number of text books to be sold in order to maximize the revenue.

Solution: For the revenue function $R(x) = -0.06x^2 + 300x$, the derivative function is $R'(x) = -0.12x + 300$ and the critical number on the interval $[0, 5000]$ is $x = 2500$.

Then evaluating the revenue function at $x = 0, x = 2500$ and $x = 5000$,and comparing the values, we have that $R(0) = 0, R(2500) = 37500$ and $R(5000) = 0$ where 2500 is the largest value of the revenue function.

$\therefore$ 2500 books maximize the revenue function and the largest revenue obtanined from this sell is \$37500.

GOOD LUCK!