SPRING 2009 PREVIEW PROBLEMS & SOLUTIONS FOR MAT 210 EXAM 1:

1. For : $f(x) = 7x + 9$, evaluate: $\lim_{x\to -2} f(x)$

   Solution:

   As $f(x) = 7x + 9$ is a polynomial function, we use direct substitution to evaluate the limit:

   $$\therefore \lim_{x\to -2} f(x) = \lim_{x\to -2} (7x + 9) = 7\,(-2) + 9 = -14 + 9 = 5$$

2. Given the function : $f(x) = x^2 + 5$. Find $\lim_{x\to 4} \frac{f(x)-f(4)}{x-4}$:

   Solution:

   $$\lim_{x\to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x\to 4} \frac{x^2 + 5 - 21}{x - 4} = \lim_{x\to 4} \frac{x^2 - 16}{x - 4}$$

   $$= \lim_{x\to 4} \frac{(x - 4)\,(x + 4)}{x - 4} = \lim_{x\to 4} (x + 4) = 8$$

3. Evaluate: $\lim_{h\to 0} \frac{f(z+h)-f(z)}{z}$: where $\frac{\left(\frac{1}{z+h} - \frac{1}{z}\right)}{h}$ is the difference quotient of the function : $f\,(z) = \frac{1}{z}$

   Solution:

   $$\lim_{h\to 0} \frac{f(z + h) - f(z)}{z} = \lim_{h\to 0} \frac{\left(\frac{1}{z+h} - \frac{1}{z}\right)}{h}$$

   $$= \lim_{h\to 0} \frac{1}{h} \left(\frac{z - (z + h))}{z\,(z + h)} \right) = \lim_{h\to 0} \frac{-h}{hz(z + h)} = \lim_{h\to 0} \frac{-1}{z(z + h)} = \frac{-1}{z^2}$$

4. Given the function : $g(x) = 3x^2 + 5$, and $a = 7$.

   Evaluate each of the following limits (if exists):

   (a) (Right Hand Limit: RHL) $\lim_{x\to a^+} g(x)$

   Solution:

   $$\lim_{x\to a^+} g(x) = \lim_{x\to 7^+} \left(3x^2 + 5\right) = 152$$

   (b) (Left Hand Limit: LHL) $\lim_{x\to a^-} g(x)$

   Solution:

   $$\lim_{x\to a^-} g(x) = \lim_{x\to 7^-} \left(3x^2 + 5\right) = 152$$

(c) Does $\lim_{x\to a} g(x)$ exist? Yes .

As the RHL and the LHL of g at $a = 7$ are the same, the common limit is what the limit of g at $a = 7$.

That is,

$$\lim_{x\to 7} g(x) = 152$$

5. For the function : $k(x) = \begin{cases} 6x + \lambda, \text{ for } x > 2 \\ -x + 15, \text{ for } x < 2 \end{cases}$

(a) Determine the value of λ so that the function k has limit at $a = 2$.

Solution:

We say the limit at $a = 2$ exists when both the RHL and LHL of k exist there and are equal.

Thus

$$RHL : \lim_{x\to 2^+} k(x) = \lim_{x\to 2^+} (6x + \lambda) = 12 + \lambda$$

and

$$LHL : \lim_{x\to 2^-} k(x) = \lim_{x\to 2^-} (-x + 15) = -2 + 15 = 13.$$

Then equalizing :

$$RHL = LHL$$

, we have :

$$12 + \lambda = 13.$$

$$\therefore \lambda = 1$$

6. Identify the points where the function: $f(x) = \frac{3x+1}{x^2+3x+2}$ is discontinuous?

Solution: The given function f is a rational function and we have learnt that rational functions are continuous at every point of their domains.

For this function the domain is the set of real numbers where the denominator:

$$x^2 + 3x + 2 \neq 0 :$$

which is the same as saying:

$$(x + 2)(x + 1) \neq 0 \Leftrightarrow x \neq -2, -1$$

$$\therefore f \quad \text{is discontinuous at } a = -2 \text{ and } a = -1$$

7. For the function : $f(x) = \frac{3x^2+5x+1}{6x^2+1}$,
by evaluating the limit at infinity : $\lim_{x\to\infty} f(x) =: f(\infty)$, find the horizontal asymptote to the graph of f.

Solution: From the fact that : $\frac{1}{x} \to 0$ as $x \to \infty$ and $\frac{1}{x^2} \to 0$ as $x \to \infty$ we evaluate :

$$f(\infty) = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x^2+5x+1}{6x^2+1} = \lim_{x\to\infty} \frac{x^2\left(3+\frac{5}{x}+\frac{1}{x^2}\right)}{x^2\left(6+\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\left(3+\frac{5}{x}+\frac{1}{x^2}\right)}{\left(6+\frac{1}{x^2}\right)} = \frac{3+0+0}{6+0} = \frac{3}{6} = \frac{1}{2}$$

$\therefore y = \frac{1}{2}$ is the horizontal assymptote to the graph of f

8. **Application of limit at infinity**: The cost of producing x number of units of a certain product per month is given by:
$C(x) = 450x + 3500$, and the average cost of producing x units of the product is given by : $C_{ave}(x) = \frac{C(x)}{x}$. Find

   (a) $C_{ave}(10)$: average cost of producing 10 units of the product.

   (b) The terminal average cost : $C_{ave}(\infty) := \lim_{x\to\infty} C_{ave}(x)$

   Solution: a)

$$\begin{aligned} C_{ave}(10) &= \frac{C(10)}{10} \\ &= \frac{450(10)+3500}{10} = \frac{8000}{10} = 800 \end{aligned}$$

$$\therefore C_{ave}(10) = \$800^{\underline{00}}$$

b) Terminal average cost :$C_{ave}(\infty)$ is calculated as :

$$\begin{aligned} C_{ave}(\infty) &= \lim_{x\to\infty} C_{ave}(x) \\ &= \lim_{x\to\infty} \frac{C(x)}{x} \\ &= \lim_{x\to\infty} \frac{450x+3500}{x} \end{aligned}$$

$$= \lim_{x\to\infty} \frac{x\left(450+\frac{3500}{x}\right)}{x}$$

$$= \lim_{x\to\infty} \frac{\left(450+\frac{3500}{x}\right)}{1} = \lim_{x\to\infty} \left(450+\frac{3500}{x}\right) = 450$$

$$\therefore C_{ave}(\infty) = \$450.00$$

Note: $C_{ave}(\infty)$ is the slope of the linear cost function.

9. A vehicle is moving linearly and its distance S as a function of time t in secs is given by : $S(t) = 3t^2 + 10$ where S is in miles.

   a) compute the average velocity of the vehicle between the time of 3 secs and 5 secs.

   b) compute the instantaneous velocity of the vehicle when $t = 3$ secs: $v(3) := \lim_{h\to 0} \frac{S(3+h)-S(3)}{h}$

   Solution: a) The average velocity of the vehicle on the time interval :$[3\,\text{sec}, 5\,\text{sec}]$ is computed as :

$$\begin{aligned} \frac{S(5) - S(3)}{5-3} &= \frac{3(5)^2 + 10 - \left(3(3)^2 + 10\right)}{2} \\ &= \frac{85-37}{2} = \frac{48}{2} = 24 \end{aligned}$$

$\therefore$ The average velocity is 24 miles/sec.

b) The instantaneous velocity $v(3) = S'(3)$ at $t = 3\,\text{sec}$ is computed as :

$$\begin{aligned} v(3) &= \lim_{h\to 0} \frac{S(3+h) - S(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2 + 10 - \left(3(3)^2 + 10\right)}{h} \\ &= \lim_{h\to 0} \frac{3\left(9 + 6h + h^2\right) + 10 - 37}{h} \end{aligned}$$

$$\begin{aligned} &= \lim_{h\to 0} \frac{18h + 3h^2}{h} \\ &= \lim_{h\to 0} \frac{h(18+3h)}{h} = \lim_{h\to 0} (18 + 3h) = 18 \end{aligned}$$

The instantaneous velocity at $t = 3\,\text{sec}$ is just the derivative of the distance function at $t = 3\,\text{sec}$.

$\therefore S'(3) = v(3) = 18$ miles/sec.

10. For the function $f(x) = 4x^2 - 1$, write the equation of the tangent line to the graph at $(1, 3)$.

Solution:

The derivative of the function at $a = 1$:$f'(1)$ is the slope of the tangent line to the graph of f

That is:

$$m_{\tan gent} = f('1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h\to 0} \frac{4(1+h)^2 - 1 - (4(1)^1 - 1)}{h}$$

$$= \lim_{h\to 0} \frac{4\left(1 + 2h + h^2\right) - 1 - (4-1)}{h}$$
$$= \lim_{h\to 0} \frac{8h + 4h^2}{h}$$
$$= \lim_{h\to 0} (8 + 4h) = 8$$

And the line passes through the point of contact : $(1, 3)$. Then from slope point form of equation of a straight line :

we have the equation of the tangent line to be :

$$y - y_0 = m(x - x_0)$$

Then putting every thing in place :

$$y - 3 = 8(x - 1)$$

and simplifying this we have the equation of the tangent line $\ell_{\tan gent}$ to be :

$$\ell_{\tan gent} : y = 8x - 5$$

Sketch both graphs of f and the tangent line $\ell_{\tan gent}$:

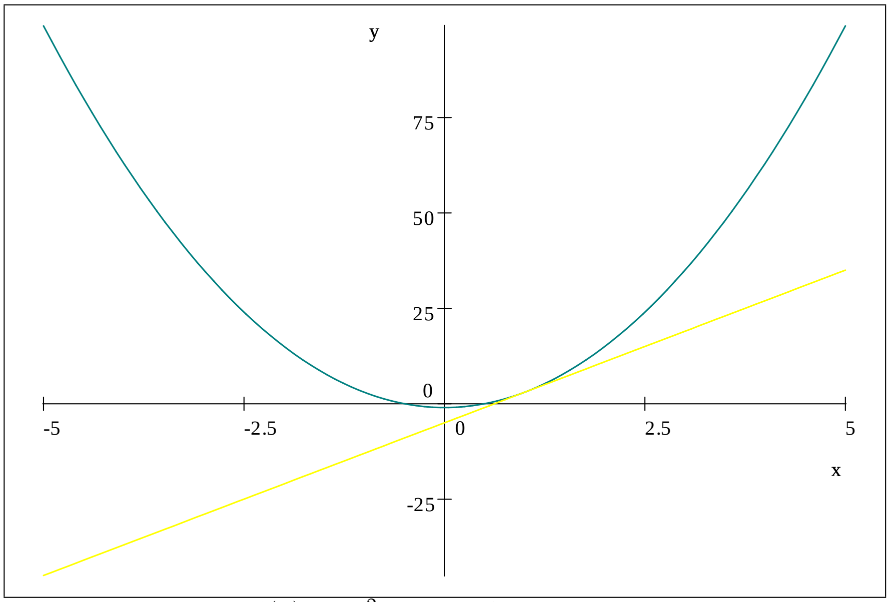

Fig. $f(x) = 4x^2 - 1$ and $\ell_{\tan} : y = 8x - 5$