Problems by Dejenie Alemayehu Lakew

Problem 1 *Define the function f over the domain $\Omega = [0, 2]$ by*

$$f(x) = \begin{cases} x, \text{ for } 0 \leq x \leq \alpha \\ 1, \text{ for } \alpha < x \leq 2 \end{cases}$$

and g be the function defined by

$$g(x) = \begin{cases} 1, \text{ for } 0 < x < \alpha \\ 0, \text{ for } \alpha < x < 2 \end{cases}$$

Prove that g is a first order distributional derivative of f over Ω only if $\alpha = 1$.

Proof. Let $v \in C_0^\infty(\Omega)$ be an arbitrary function. Then we have to check the equality:

$$\int_\Omega gv dx = -\int_\Omega fv' dx.$$

But then

$$\begin{aligned} \int_\Omega fv' dx &= \int_0^\alpha xv' dx + \int_\alpha^2 v' dx \\ &= (\alpha - 1) v(\alpha) - \int_0^\alpha v dx \end{aligned}$$

and

$$\int_\Omega gv dx = \int_0^\alpha v dx.$$

Therefore from the definition of a distributional or week derivative of first order: $g = D_w f$ over Ω if

$$\int_\Omega gv dx = -\int_\Omega fv' dx, \ \forall v \in C_0^\infty(\Omega).$$

That is

$$-\int_0^\alpha v dx = (\alpha - 1) v(\alpha) - \int_0^\alpha v dx$$

which implies that

$$\begin{aligned} (\alpha - 1) v(\alpha) &= 0 \\ &\Longrightarrow \text{ either } \quad \alpha = 1 \quad \text{or} \quad v(\alpha) = 0. \end{aligned}$$

But from the arbitraryness of v, we have $\alpha = 1$.

That means the function g has to be continuous at $x = \alpha = 1$ ■

Problem 2 *For the function*

$$f(x) = \begin{cases} \alpha x, \text{ for } 0 \leq x \leq \beta \\ 1, \text{ for } \beta < x \leq 2 \end{cases}$$

prove that

$$g(x) = \begin{cases} \alpha, \text{ for } 0 < x < \beta \\ 1, \text{ for } \beta < x < 2 \end{cases}$$

is a first order distributional derivative of f over $\Omega = [0, 2]$

Proof. Again, for $v \in C_0^\infty(\Omega)$, we have to impose the equality:

$$\int_\Omega f v' dx = -\int_\Omega g v dx.$$

But then

$$\begin{aligned} \int_\Omega f v' dx &= \int_0^\beta x v' dx + \int_\beta^2 v' dx \\ &= v(\beta)(\alpha\beta - 1) - \alpha \int_0^\beta v dx \end{aligned}$$

and

$$-\int_\Omega g v dx = -\alpha \int_0^\beta v dx.$$

Thus

$$\begin{aligned} \alpha \int_0^\beta v dx &= v(\beta)(\alpha\beta - 1) - \alpha \int_0^\beta v dx \\ &\iff \quad \alpha\beta - 1 = 0 \quad \text{or} \quad v(\beta) = 0. \end{aligned}$$

From the arbitrayness of v, we have

$$\alpha\beta - 1 = 0.$$

That is

$$\alpha\beta = 1 \quad \text{or} \quad \alpha = \beta^{-1}.$$

Again, here the function f has to be continuous at $x = \beta$ which is when $\alpha\beta = 1$ in order it has a weak or distributional derivative over Ω. ■