✠ : Preview problems and solutions to MATH 210 Exam 3:

Students: Study the procedures carefully. Good Luck! .

1. Differentiate the following functions:

   a) $f(x) = e^{3x+5}$

   b) $g(x) = xe^{x^2}$

   c) $h(x) = 3^x$

Solution 1

a) $f'(x) = \left(e^{3x+5}\right)' \underset{\substack{\uparrow \\ chain\ rule}}{=} e^{3x+5}\,(3x+5)' = 3e^{3x+5}$

b) $g'(x) = \left(xe^{x^2}\right)' \underset{\substack{\uparrow \\ product\ and\ chain\ rules}}{=} 1.e^{x^2} + x.2x.e^{x^2}$

$$= e^{x^2} + 2x^2e^{x^2}$$

c) $h'(x) = (3^x)' \underbrace{=}_{change\ of\ base} \left(e^{x\ln 3}\right)'$

$$\underbrace{=}_{chain\ rule} e^{x\ln 3}\,(x\ln 3)' = e^{x\ln 3}\ln 3 = 3^x\,(\ln 3)$$

2. Differentiate the following functions:

   a) $f(x) = \ln\left(x^2 + 12\right)$

   b) $g(x) = \log_5\left(3x^2 + 1\right)$

Solution 2 *(Use chain rule and the following rules for differentiating logarithmic functions:)*

(1): $(\ln(f(x))' = \frac{1}{f(x)}.f'(x) = \frac{f'(x)}{f(x)}$

(2): $(\log_a f(x))' = \left(\frac{\ln f(x)}{\ln a}\right)' = \frac{1}{\ln a}\,(\ln f(x))'$

$$= \frac{1}{\ln a}\left(\frac{1}{f(x)}.f'(x)\right)$$

$$= \frac{f'(x)}{(\ln a)f(x)}$$

The solutions then are :

a) $f'(x) = \left(\ln\left(x^2+12\right)\right)' = \frac{1}{x^2+12}\left(2x\right) = \frac{2x}{x^2+12}$

b) $g'(x) = \left(\log_5\left(3x^2+1\right)\right)'$

$$= \left(\frac{\ln\left(3x^2+1\right)}{\ln 5}\right)'$$
$$= \frac{1}{\ln 5}\left(\ln\left(3x^2+12\right)\right)'$$
$$= \frac{1}{(\ln 5)(3x^2+12)}\left(6x\right)$$
$$= \frac{6x}{(3\ln 5)x^2+12\ln 5}$$

3. For the composition function: $h(x) = g(f(x))$,
   if $f(1) = 5, f'(1) = 6$ and $g'(5) = 10$, then what is : $h'(1)$

Solution 3
$h(x) = g(f(x)) \Rightarrow h'(x) \underbrace{=}_{chain\ rule} g'(f(x)).f'(x)$

Then , when $x = 1$, we have:

$h'(1) = g'(f(1)).f'(1) = g'(5) \times 6 = 10 \times 6 = 60$

4. Find the equation of the tangent and normal lines to the graph of the function:
   $y = e^{2x+1}$ at the point $(0, e)$

Solution 4 *Tangent line :*

$y' = f'(x) = \left(e^{2x+1}\right)' = e^{2x+1} \times (2) = 2e^{2x+1}.$

Then slope of the tangent line at (0,1) is : $f'(0) = 2e^{2(0)+1} = 2e.$

Using point-slope form :$y - y_0 = m\left(x - x_0\right)$ *with* $(x_0, y_0) = (0, e)$ *and* $m = 2e$*, we get the equation of the tangent line :*

$y - e = 2e\left(x - 0\right) = \left(2e\right)x$

$\therefore l_{\tan gent} : y = (2e)\,x + e$

Normal line:

The line that is perpendicular to the tangent line and passes through the same point of contact is called the normal line.

Since the two lines are perpendicular, the slopes are negative reciprocals of each other and hence the normal line has slope: $\frac{-1}{2e}$.

using again slope-point form : the normal line has equation given by:

$y - e = \frac{-1}{2e}(x - 0) = \frac{-1}{2e}x \Rightarrow y = \frac{-1}{2e}x + e$

$\therefore l_{normal} : y = \frac{-1}{2e}x + e$

Multiple Questions: Choose the correct answer.

5. Which of the following is true about the function: $f(x) = x^2 + 1$?

   a. $f'(x) = 2x$ **b.** $f'(2) = 4$ **c.** $f(2) = 5$ **d.** All are true.

   Answer: (**d**)

   Reason: (**a**) and (**c**) is trivially true. For (**b**) : $f'(2) = f'(x)_{|x=2} = 2x_{|x=2} = 4$.

6. What is the derivative $f'(x)$ of : $f(x) = x\,(x+2)^5$: use product and chain rule:

   a. $f'(x) = (6x+2)(x+2)^4$ **b.** $f'(x) = (x+2)^5 + 5(x+2)^4$ **c.** $f'(x) = (5x+2)(x+1)^4$ **d.** $f'(x) = 6x^5 + 25$

   Answer: (**a**)

   Reason: $f'(x) = \frac{d}{dx}\left[x\,(x+2)^5\right] \underbrace{=}_{\text{product rule}} (x)'\,(x+2)^5 + x\left((x+2)^5\right)'$

   $$= 1\,(x+2)^5 + x\left(\underbrace{5\,(x+2)^4}_{\text{chain rule}}\right)$$

   $$= (x+2)^5 + 5x\,(x+2)^4 = (x+2)^4\,(x+2+5x)$$

   $$= (6x+2)\,(x+2)^4$$

7. The equation of the tangent line to the graph of: $f(x) = 3x^2$ at the point : $(1,3)$ is :

a. $y = 6x - 3$ **b.** $y = 6x + 3$ **c.** $y = 3x - 3$ **d.** $y = 3x + 1$

Answer: (**a**)

Reason: First , compute the slope which is the derivative of the function at: $a = 1$

Slope: $m = f'(1) = \underbrace{f'(x)_{|x=1}}_{\text{evaluating } f'(x) \text{ at } x=1} = 6x_{|x=1} = 6$

Then from slope-point form: with $m = 6$ and point : $\left(\underset{x_1}{\underset{\uparrow}{1}}, \underset{y_1}{\underset{\uparrow}{3}}\right)$ the equattion of the tangent line is:

$y - 3 = 6(x - 1) = 6x - 6 \Rightarrow y = 6x - 3$

$\therefore$ $y = 6x - 3$ is the equation of the tangent line.

$\therefore$ (**a**) is the correct answer.

8. Which of the following is correct about the function: $f(x) = \frac{x}{\sqrt{x^2+1}}$:

a. $f'(x) = \frac{1}{x^2+1}$ **b.** $f''(x) = \frac{-2x}{(x^2+1)^2}$ **c.** $f''(1) = \frac{-1}{2}$ **d.** All are true

Answer: (**d**)

Reason : $f(x) = \frac{x}{\sqrt{x^2+1}}$, then the first derivative is:

$$f'(x) = \left(\frac{x}{\sqrt{x^2+1}}\right)' \underset{\text{quotient rule}}{\underset{\uparrow}{=}} \frac{(x)'\left(\sqrt{x^2+1}\right) - x\left(\sqrt{x^2+1}\right)'}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{1\left(\sqrt{x^2+1}\right) - x\frac{2x}{2\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2} = \frac{1\left(\sqrt{x^2+1}\right) - \frac{x^2}{\sqrt{x^2+1}}}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{x^2+1-x^2}{x^2+1} = \frac{1}{x^2+1} \text{ which is } (\mathbf{a}).$$

Next the second derivative of f :

$$f''(x) = (f'(x))' = \frac{d}{dx}\left(\frac{1}{x^2+1}\right)$$

$$\underset{\text{quotient rule}}{\underset{\uparrow}{=}} \frac{\overset{\text{derivative of a constant}}{\overset{\downarrow}{(1)'}} \quad (x^2+1) - 1(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{-(2x)}{(x^2+1)^2} \text{ which is } (\mathbf{b}).$$

Then finally, $f''(1) = f''(x)_{|x=1} = \left(\frac{-(2x)}{(x^2+1)^2}\right)_{|x=1}$

$$= \frac{-2}{(1^2+1)^2} = \frac{-2}{4} = \frac{-1}{2} \text{ which answers } (\mathbf{c})$$

$\therefore$ (**d**) is the correct answer.