MATH 427 EXAM 3 SPRING 2009

Solutions to the problems

1. For the IVP : $\frac{dy}{dx} = ax + by$ with IC: $y(0) = 0$

   a) Show that the solution is given by :

$$y(x) = \frac{a}{b^2}\left(e^{bx} - bx - 1\right)$$

   b) when $b < 0$, which part of the solution is dominant and which part goes to zero as $x \to \infty$ (i.e., transient)

   Solution:

   a) The DE:

$$y' = ax + by$$

   is written in the form:

$$y' - by = ax .... (\star)$$

   which is a linear DE with $p(x) = -b$ and $f(x) = ax$

   The integrating factor then is :

$$\mu(x) = e^{\displaystyle\int p(x)dx} = e^{\displaystyle\int -bdx} = e^{-bx}.$$

   Multiplying both sides of the DE $(\star)$ by $\mu(x)$ we get:

$$\underbrace{(y' - by)\, e^{-bx}}_{\overset{\|}{\frac{d}{dx}(ye^{-bx})}} = ax\left(e^{-bx}\right) = axe^{-bx}$$

$$\Rightarrow d\left(ye^{-bx}\right) = axe^{-bx}dx.$$

   Integrating both sides of the equality, we have :

$$\int d\left(ye^{-bx}\right) = \int axe^{-bx}dx.$$

   But from integration by parts:

$$\int axe^{-bx}dx = \frac{-a}{b}xe^{-bx} - \frac{a}{b^2}e^{-bx} + c_1$$

   and

$$\int d\left(ye^{-bx}\right) = ye^{-bx} + c_2$$

Therefore, combining the two results we have :

$$ye^{-bx} = c + \frac{-a}{b}xe^{-bx} - \frac{a}{b^2}e^{-bx}$$

Solving for y we have

$$y(x) = e^{bx}\left(c + \frac{-a}{b}xe^{-bx} - \frac{a}{b^2}e^{-bx}\right) = ce^{bx} - \frac{a}{b}x - \frac{a}{b^2}$$

and from the IC:

$$y(0) = 0 \Rightarrow c = \frac{a}{b^2}$$

$\therefore$ The solution function is :

$$y(x) = \underbrace{\frac{a}{b^2}e^{bx}}_{\substack{\shortparallel \\ \phi(x)}} \underbrace{-\frac{a}{b}x - \frac{a}{b^2}}_{\substack{\shortparallel \\ \psi(x)}} = \frac{a}{b^2}\left(e^{bx} - bx - 1\right)$$

as required.

b) When $b < 0$, the the part of the solution function :

$$\phi(x) = e^{bx} \searrow 0 \text{ as } x \to \infty,$$

which is the transient part of the solution function.

Therefore, the dominant part of the solution will be :

$$\psi(x) = -\frac{a}{b}x - \frac{a}{b^2}.$$

2. Find the singularity of the differential equation and solve:

$$x^2\frac{dy}{dx} + xy = x^2e^x$$

Solution:

The singularity of the DE is: $x = 0$. Then solution will be defined either in $(-\infty, 0)$ or in $(0, \infty)$ depending on the initial condition.

then dividing both sides of the equation:

$$x^2\frac{dy}{dx} + xy = x^2e^x$$

by x^2, we have :

$$y' + \frac{1}{x}y = e^x \ldots (\circledast).$$

Thus, $p(x) = \frac{1}{x}$, and choosing $x > 0$, the integrating factor is:

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x.$$

Multiplying both sides of $(\circledast)$ by the integrating factor $\mu(x)$, we have :

$$\left(y' + \frac{1}{x}y\right) \underset{\mu(x)}{\underset{\uparrow}{x}} = (e^x) \underset{\mu(x)}{\underset{\uparrow}{x}} = xe^x$$

and

$$\left(y' + \frac{1}{x}y\right) x = \frac{d}{dx}(yx) \Rightarrow \frac{d}{dx}(yx) = xe^x$$

Thus $d(yx) = xe^x dx$ and integrating both sides we have:

$$\underbrace{\int d(yx)}_{\substack{\| \\ yx}} = \int xe^x dx = xe^x - e^x + c$$

That is,

$$yx = xe^x - e^x + c.$$

Solving for y we have

$$y(x) = e^x - \frac{1}{x}e^x + \frac{c}{x}$$

which is the required solution.

3. For the differential equation:

$$(e^{xy} + xye^{xy} + y)\,dx + \left(x^2e^{xy} + x\right) dy = 0$$

(a) Check for exactness

(b) Find a solution

Solution:

a) Exactness: $M(x, y) = e^{xy} + xye^{xy} + y$ and $N(x, y) = x^2e^{xy} + x$. Computing the two partial derivatives:

$$M_y(x, y) = 2xe^{xy} + yx^2e^{xy} + 1$$

and

$$N_x(x, y) = 2xe^{xy} + yx^2e^{xy} + 1$$

we have : $M_y(x, y) = N_x(x, y)$ which implies that the DE is exact.

b) Retrieve the solution function as usual :

$$f(x,y)=\int M(x,y)dx+g(y)=\int\left(e^{xy}+xye^{xy}+y\right)dx+\underset{\text{Integrating constant}}{\underset{\uparrow}{g(y)}}$$

$$=\frac{1}{y}e^{xy}+xe^{xy}-\frac{1}{y}e^{xy}+yx+g(y)=xe^{xy}+xy+g(y)$$

Then differentiating $f(x,y)$ with repect to y we have:

$$f_y\left(x,y\right)=\frac{\partial}{\partial y}\left(xe^{xy}+xy+g(y)\right)=x^2e^{xy}+x+g'(y)\underset{\text{due to exactness}}{\underset{\uparrow}{=}}N(x,y)=x^2e^{xy}+x$$

$$\Rightarrow g'(y)=0$$

$$\therefore g(y)=cons\tan t$$

Hence the solution function is :

$$f(x,y)=xe^{xy}+xy=c$$

4. For the differential equation: $\left(3x+2y^2\right)dx+\left(2xy\right)dy=0$
   a) Show that the DE is not exact
   b) Compute an integrating factor
   c) Solve the DE

Solution:

a) Checking exactness: $M(x,y)=3x+2y^2$ and $N(x,y)=2xy$. Then

$$M_y\left(x,y\right)=\left(3x+2y^2\right)_y=4y \text{ and } N_x\left(x,y\right)=\left(2xy\right)_x=2y$$

Hence,

$$M_y\left(x,y\right)\neq N_x\left(x,y\right).$$

Therefore the DE is not exact.

b) We then search for integrating factor : $\mu\left(x,y\right)$ so that

$$\left(\mu M\right)dx+\left(\mu N\right)dy=0$$

is exact.

That is

$$\left(\mu M\right)_y=\mu_y M+\mu M_y=\left(\mu N\right)_x=\mu_x N+\mu N_x$$

Assuming μ as a function of y only, works the job and we have : $\mu_y = 0$. Therefore,

$$\mu_y M + \mu M_y = (\mu N)_x = \mu_x N + \mu N_x,$$

yields

$$\mu M_y = \mu_x N + \mu N_x \Rightarrow \mu_x N = \mu M_y - \mu N_x = \mu \left(M_y - N_x\right).$$

That is

$$\frac{d\mu}{dx} = \frac{\mu \left(M_y - N_x\right)}{N} = \frac{\mu \left(4y - 2y\right)}{2xy} = \frac{\mu}{x}$$

which is a separable DE. Thus, $\frac{d\mu}{\mu} = \frac{1}{x}dx$. Integrating both sides we have :

$\int \frac{d\mu}{\mu} = \int \frac{dx}{x} \implies \ln|\mu| = \ln|x|$. Take $\mu(x) = x$ and it works fine. Therefore, the new DE:

$$\underbrace{x\left(3x + 2y^2\right)dx + x\left(2xy\right)dy = 0}_{\substack{\Downarrow \\ (3x^2+2xy^2)dx+(2x^2y)dy=0}}$$

is exact.

c) Then we retrive the solution function as :

$$f(x,y) = \int \left(3x^2 + 2xy^2\right)dx + g(y) = x^3 + x^2y^2 + g(y).$$

Then

$$\frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial y}\left(x^3 + x^2y^2 + g(y)\right) = 2x^2y + g'(y) = 2x^2y = \text{the new } N(x,y).$$

Thus,

$$g'(y) = 0 \Rightarrow g(y) = \text{cons}\tan\text{t}.$$

$\therefore$The solution function is given by :

$$f(x,y) = x^3 + x^2y^2 = c$$

5. Solve the Bernoulli's differential equation:

$$y' + \frac{1}{x}y = xy^{-2}\text{.....}\,(\bigstar)$$

Solution:

Here $n = -2$. Then use the substitution: $u = y^{1-(-2)} = y^3$. Then $u' = 3y^2y'$. Multiplying both sides of the DE: (★) by $3y^2$ we have :

$$\left(3y^2\right) y' + \frac{1}{x}\left(3y^2\right) y = x\left(3y^2\right) y^{-2}$$
$$\Rightarrow u' + \frac{1}{x} u = 3x$$

which a linear equation with $p(x) = \frac{1}{x}$ and therefore the integrating factor is :

$$\mu(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x, \text{ for } x > 0$$

Therefore multiplying both sides with the integrating factor, we have :

$$\left(u' + \frac{1}{x}u\right) \underset{\mu(x)}{\underset{\uparrow}{x}} = (3x) \underset{\mu(x)}{\underset{\uparrow}{x}} \Leftrightarrow \frac{d}{dx}(ux) = 3x^2$$

Then $d(ux) = 3x^2dx$, integrating both sides of the equation,

$$\int d(ux) = \int 3x^2dx \Rightarrow ux = x^3 + c$$

$\therefore u = x^2 + \frac{c}{x}$. But $u = y^3$. Thus,

$$y = \sqrt[3]{x^2 + \frac{c}{x}}$$

is the required solution.

6. A particle is moving in the $x - y$ plane and its motion is ruled by the differential equation: $y' = \sin^2(x - y + 5)$. Determine the equation of path of the particle, if it starts its travel at point $(1, 1)$.

Solution:

Here the differential equation of motion is :

$$y' = \sin^2(x - y + 5)$$

and begining point is : $(1, 1)$. Using a substituition :

$$u = x - y + 5$$

we have :

$$u' = 1 - y' = 1 - \sin^2 u = \cos^2 u$$

which is a separabel DE in variable u. Therefore

$$\frac{du}{\cos^2 u} = dx \Rightarrow \sec^2 u du = dx.$$

Integrating both sides, we have :

$$\int \sec^2 u du = \int dx \Rightarrow x = \tan u + c \Leftrightarrow u = \arctan\left(x + c\right).$$

Putting $u = x - y + 5$ in place and using the initial condition, we have :

$$y(x) = x - \arctan(x + \tan 5 - 1) + 5$$

as the solution function.

7. Solve : $y' = 1 + e^{-x+y+12}$, $y\left(1\right) = 4$

Solution:

Here use the substituition :

$$u = -x + y + 12$$

to change the DE to a separable one :

$$u' = -1 + y' = -1 + 1 + e^u = e^u$$

Thus $\frac{du}{e^u} = dx \Leftrightarrow e^{-u}du = dx$. Integrating both sides, we have : $e^{-u} = c - x$ and solving for u we have :

$$u = -\ln\left(c - x\right)$$

and using the initial conditions we have :

$$y = x + \ln\left(1 + e^{15} - x\right) - 12$$

Note : The maximum interval of existence $I_{\max}$ for the solution curve is determined from the domain of the integral : $1 + e^{15} - x > 0$.

$\therefore$ $I_{\max} = \left(-\infty, 1 + e^{15}\right)$. If x has to be non-negative (in case the independent variable is time t) , then $I_{\max} = [0, 1 + e^{15})$

8. Given the differential equation : $ydx + \left(-x + \sqrt{xy}\right) dy = 0$

a) Show that it is homogeneous of degree 1

b) Solve the DE

Solution:

a) Checking for homogeniety: $M(x,y)=y$ and $N(x,y)=-x+\sqrt{xy}$ and for $t>0$, we have :

$$M\,(tx.ty)=ty=tM(x,y)$$

and

$$N(tx,ty)=-tx+\sqrt{txty}=-tx+\sqrt{t^2xy}=-tx+t\sqrt{xy}=t\,(-x+\sqrt{xy})=tN(x,y).$$

We see that M and N are homogeneous functions of degree 1.

b) use the substituition: $y=ux$ .Thus

$$dy=udx+xdu$$

and

$$ydx+(-x+\sqrt{xy})\,dy=0$$

will be:

$$uxdx+\left(-x+\sqrt{xux}\right)(udx+xdu)=0$$

which implies :

$$\left(xu\sqrt{u}\right)dx+x^2\left(-1+\sqrt{u}\right)du=0\Leftrightarrow u\sqrt{u}dx+x\left(-1+\sqrt{u}\right)du=0$$

which is separable.

Thus we have :

$$\frac{dx}{x}+\frac{(-1+\sqrt{u})}{u\sqrt{u}}du=0\Rightarrow\frac{dx}{x}=\left(\frac{1}{u\sqrt{u}}-\frac{1}{u}\right)du$$

integrating both sides

$$\begin{aligned}\int\frac{dx}{x}&=\int\left(\frac{1}{u\sqrt{u}}-\frac{1}{u}\right)du\\ ye^{\frac{2}{\sqrt{(y/x)}}}&=c\end{aligned}$$

is an implicit one parameter family of solutions.