A New Look at Higher Order Derivatives

By

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Computing higher order derivatives of a differentiable function is a several steps process that we know from elementary calculus. In this note, we give a theorem which is an analogue of a definition of the derivative we know interms of a diffence quotient, for higher order derivatives. We give an example later to see the validity of the theorem. First, a first order analouge of a difference quotient for higher order is defined and a theorem of n^{th}-order derivative interms of the n^{th}-order difference quotient is presented.

Definition 1 *The n^{th}-order difference quotient of a function ψ is defined as :*

$$h^{-n}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-1)^{k}\psi(x_0+(n-k)h)\right)=\frac{\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-1)^{k}\psi(x_0+(n-k)h)}{h^{n}}$$

where $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ is a binomial coefficient.

Theorem 2 *(Dejenie A. Lakew) Let Ω be a non-empty open subset of $\mathbb{R}$ and $\psi \in C^{(n+1)}(\Omega, \mathbb{R})$. Then for $x_0 \in \Omega$ and $\forall n \in \mathbb{N}$, the $n^{th}-$order derivative of ψ at x_0 denoted by $\psi^{(n)}(x_0)$ is given by a single expression:*

$$\psi^{(n)}(x_0)=\lim_{h\to 0}h^{-n}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)(-1)^{k}\psi(x_0+(n-k)h)\right)$$

In particular, the second derivative of ψ at x_0 therefore is given by :

$$\psi^{(2)}(x_0) = \lim_{h\to 0} h^{-2}\left(\sum_{k=0}^{2}\left(\begin{array}{c}2\\k\end{array}\right)(-1)^k\,\psi(x_0+(2-k)h)\right)$$
$$= \lim_{h\to 0}\left(\frac{\psi(x_0+2h)-2\psi(x_0+h)+\psi(x_0)}{h^2}\right)$$

Example 3 *For the function* $\psi(x) = x^3$*, we know that the second derivative is :* $\psi''(x) = 6x$

Then verifying the above theorem for the second order derivative we have:

$$\psi''(x) = \lim_{h\to 0}\left(\frac{(x+2h)^3 - 2(x+h)^2 + x^3}{h^2}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^3+6x^2h+12xh^2+8h^3-2x^3-6x^2h-6xh^2-2h^3+x^3}{h^2}\right)$$

$$= \lim_{h\to 0}\left(\frac{6xh^2+6h^3}{h^2}\right)$$

$$= \lim_{h\to 0}\left(\frac{h^2(6x+6h)}{h^2}\right)$$

$$= \lim_{h\to 0}(6x+6h)$$
$$= 6x$$

which is exactly we have at the begining.

This therem is very fundamental for computing higher order derivatives by just evaluating a limit of one algebraic expression that has multiplicities of the step size and involves some coefficients that are computed using binomial expansions.