ON SOME DISCRETE DIFFERENTIAL EQUATIONS

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Abstract. In this short note, we present few results on the use of the discrete Laplace transform in solving first and second order intial value problems of discrete differential equations.

In differential equations classes, we teach the Laplace transform as one of the tools available to find solutions of linear differential equations with constant coefficients.

What we do in this note is to use a noncontinuous (or discrete) Laplace transform, that generates sequential solutions which are polynomials in $\mathbb{N}$ or quotients of such polynomials.

I hope readers will find the results very interesting.

This work is just a modified version of an unpublished work that I did few years ago.

Let f be a $\mathbb{R}$-valued sequence $f : \mathbb{N} \to \mathbb{R}$, then

Definition 1. *The discrete Laplace transform of $f(n)$ is defined as*

$$\ell_d \{f(n)\}(s) := \sum_{n=0}^{\infty} e^{-sn} f(n), \text{ where } s > 0.$$

Definition 2. *The first order difference equation of a sequence $f(n)$ is defined as*

$$\Delta f(n) := f(n+1) - f(n)$$

Proposition 1. *The first order discrete IVP:*

$$\triangle f(n) = n, f(1) = 1$$

has solution given by

$$f(n) = 1 + \frac{n^2 - n}{2}.$$

Proof. Taking the transform of both sides of the equation :
$\ell_d \{\triangle f(n)\}(s) = \ell_d \{n\}$, we get

$$(e^s - 1) \ell_d \{f(n)\}(s) - f(1) = \frac{e^s}{(e^s - 1)^2}.$$

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Substituting the value $f(1) = 1$, and simplifying the expression we get,

$$\ell_d \{f(n)\}(s) = \frac{1}{e^s - 1} + \frac{e^s}{(e^s - 1)^3}.$$

Then taking the inverse transform we have the solution :

$$f(n) = \ell_d^{-1} \left\{ \frac{1}{e^s - 1} + \frac{e^s}{(e^s - 1)^3} \right\}$$
$$= 1 + (n * 1) = 1 + \frac{n^2 - n}{2}.$$

Here "$*$ " is the convolution operator.

□

Proposition 2. *The second order IVP:*

$$\triangle^2 f(n) = n, f(1) = 1, \triangle f(1) = 2,$$

has solution given by :

$$f(n) = 2n - 1 + \frac{n(n-1)(n-2)}{6}.$$

Proof. First,

$$\triangle^2 f(n) = \triangle(\triangle f(n)) = f(n+2) - 2f(n+1) + f(n)$$

and using the initial conditions we get:

$$\ell_d \left\{\triangle^2 f(n)\right\}(s) = \left(e^{2s} - 2e^s + 1\right) \ell_d \{f(n)\}(s) - e^s - 1.$$

$$\Rightarrow (e^s - 1)^2 \ell_d \{f(n)\}(s) - e^s - 1 = \frac{e^s}{(e^s - 1)^2}$$

$$\Rightarrow \ell_d \{f(n)\}(s) = \frac{1}{(e^s - 1)^2} + \frac{e^s}{(e^s - 1)^2} + \frac{e^s}{(e^s - 1)^4}.$$

Then taking the inverse transform and using convolutions, we get the solution as :

$$f(n) = (1 * 1) + n + \frac{n(n-1)(n-2)}{6}$$
$$= 2n - 1 + \frac{n(n-1)(n-2)}{6}.$$

□

Proposition 3. $\ell_d \left\{\frac{1}{n}\right\}(s) = s - \ln(e^s - 1)$ *for* $s > 0$.

Proof.

$$\ell_d \{1\}(s) = \frac{1}{e^s - 1} = \sum_{n=1}^{\infty} e^{-sn} \quad \text{for } s > 0.$$

Integrating both sides, we get

$$\ln\left(e^{s}-1\right)-s=\sum_{n=1}^{\infty}\left(-\frac{e^{sn}}{n}\right)=\ell_{d}\left\{\frac{-1}{n}\right\}(s)$$

$$\Rightarrow \ell_{d}\left\{\frac{1}{n}\right\}(s)=s-\ln\left(e^{s}-1\right).$$

□

We now present discrete IVPs whose solutions are rational sequences in n.

Proposition 4. *The discrete IVP :*

$$n\triangle f(n)=1, f(2)=2, \textit{for } n\geq 2$$

has solution given by:

$$f(n)=1+\sum_{k=1}^{n-1}\frac{1}{k}.$$

Proof. Taking the transform of both sides of the equation, we have:

$$\ell_{d}\left\{n\triangle f(n)\right\}(s)=\frac{1}{e^{s}-1}.$$

But

$$\ell_{d}\left\{n\triangle f(n)\right\}(s)=\ell_{d}\left\{nf(n+1)-nf(n)\right\}(s)$$

$$=e^{s}\ell_{d}\left\{nf(n)\right\}(s)-e^{s}\ell_{d}\left\{f(n)\right\}(s)-\ell_{d}\left\{nf(n)\right\}(s)$$

$$=\left(e^{s}-1\right)\ell_{d}\left\{nf(n)\right\}(s)-e^{s}\ell_{d}\left\{f(n)\right\}(s).$$

Thus,

$$\left(e^{s}-1\right)\ell_{d}\left\{nf(n)\right\}-e^{s}\ell_{d}\left\{f(n)\right\}=\frac{1}{e^{s}-1}.$$

Again,

$$\ell_{d}\left\{nf(n)\right\}(s)=-\frac{d}{ds}\ell_{d}\left\{f(n)\right\}(s).$$

Therefore,

$$-\left(e^{s}-1\right)\frac{d}{ds}\ell_{d}\left\{f(n)\right\}(s)-e^{s}\ell_{d}\left\{f(n)\right\}(s)=\frac{1}{e^{s}-1},$$

which is an ordinary nonhomogeneous linear differential equation of first order in s and writing it in standard form we have :

$$\frac{d}{ds}\ell_{d}\left\{f(n)\right\}(s)+\frac{e^{s}}{e^{s}-1}\ell_{d}\left\{f(n)\right\}(s)=-\frac{1}{\left(e^{s}-1\right)^{2}}$$

whose solution for $\ell_{d}\left\{f(n)\right\}(s)$ is given by $\frac{1}{e^{s}-1}-\frac{\ln(e^{s}-1)-s}{e^{s}-1}$.
Then taking the inverse transform , we have :

$$f(n) = \ell_d^{-1}\left\{\frac{1}{e^s-1} + \frac{s-\ln(e^s-1)}{e^s-1}\right\}$$

$$= 1 + \ell_d^{-1}\{s-\ln(e^s-1)\} * \ell_d^{-1}\left\{\frac{1}{e^s-1}\right\}$$

$$= 1 + \left(\frac{1}{n} * 1\right) = 1 + \sum_{k=1}^{n-1}\frac{1}{k} \text{ for } n > 1.$$

□

Proposition 5. *For $n \geq 2$, the IVP :*

$$\triangle f(n) = \frac{1}{n^2}, f(2) = 2$$

has solution given by

$$f(n) = 1 + \sum_{k=1}^{n-1}\frac{1}{k^2}.$$

Proof. Re-writing the difference equation as : $n\triangle f(n) = \frac{1}{n}$,
taking the transform of both sides and using corollary 3.3 we get

$$\frac{d}{ds}\ell_d\{f(n)\}(s) + \frac{e^s}{e^s-1}\ell_d\{f(n)\}(s) = -\frac{s-\ln(e^s-1)}{e^s-1}.$$

Again solving for $\ell_d\{f(n)\}(s)$, we have

$$\ell_d\{f(n)\}(s) = \frac{1}{e^s-1} - \frac{1}{e^s-1}\int(s-\ln(e^s-1))\,ds.$$

Then solving for $f(n)$, we have ;

$$f(n) = 1 - \ell_d^{-1}\left\{\frac{1}{e^s-1}\right\} * \ell_d^{-1}\left\{\int(s-\ln(e^s-1))\,ds\right\}$$

$$\begin{aligned} &= 1 - \left(1 * \left(-\frac{1}{n^2}\right)\right) \\ &= 1 + \sum_{k=1}^{n-1}\frac{1}{k^2} \end{aligned}$$

which is the discrete solution of the given discrete differential equation given in the proposition. □

Remark 1. *The consequences of the results in this short paper are really far reaching, if used and expanded further.*

References

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