

THE SPHERICAL $\pi_{\alpha, S^{n-1}}$ -OPERATOR

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ABSTRACT. In this article we define the spherical $\pi_{\alpha, S^{n-1}}$ operator over domains in the $(n-1)D$ -unit sphere S^{n-1} of \mathbb{R}^n and develop new and analogous results. We introduce a spherical Dirac operator $\Gamma_\alpha := \Gamma_\omega + \alpha$, where $\alpha \in \mathbb{C}$ and $\Gamma_\omega = -\omega \wedge D_\omega$, the anti-symmetric Grassmanian product of ω with $D_\omega = \sum_{i=1}^n e_i \frac{\partial}{\partial \omega_i}$. We use a Gegenbauer polynomial $\Psi_\alpha^n(\omega - v)$ as a Cauchy kernel for Γ_α .

1. Introduction

The π -operator is one of the tools used to study smoothness of functions over Sobolev spaces and to solve some first order partial differential equations such as the Beltrami equation. In Euclidean spaces, we see that the singularity of its kernel is of order one more than the dimension of the space \mathbb{R}^n and hence it is a hyper singular integral operator.

In the class of singular integral operators, the π -operator is the least studied integral operator than the weakly singular and singular operators which are studied extensively.

Recently in [8] Dejenie A. Lakew and John Ryan also study the π -operator in a generalized setting over Domain Manifolds in \mathbb{C}^{n+1} and produced some properties and its integral representation as well.

In this paper we study the $\pi_{\alpha, S^{n-1}}$ -operator over domains in S^{n-1} , the $(n-1)D$ -unit sphere in \mathbb{R}^n . The differential operator we are considering is the spherical Dirac operator

$$\Gamma_\alpha := \Gamma_\omega + \alpha$$

where

$$\Gamma_\omega = -\omega \wedge D_\omega$$

Date: November 19, 2008.

2000 Mathematics Subject Classification. 30G35, 35A20, 58J15.

Key words and phrases. Spherical Dirac Operator, Pi-Operaor, Beltrami equation.

This paper is in final form and no version of it will be submitted for publication elsewhere.

and α is some complex number. Here

$$D_\omega = \sum_{i=1}^n e_i \frac{\partial}{\partial \omega_i}$$

is the usual Dirac operator in \mathbb{R}^n and \wedge is the Grassman (or wedge) product. The function which is used as a Cauchy kernel or fundamental solution to this spherical Dirac operator is a Gegenbauer polynomial.

2. Preliminaries: Algebraic and Analytic

Let $e_1, e_2, e_3, \dots, e_n$ be orthonormal unit vectors that generate \mathbb{R}^n . Then the anti-commutative algebra of dimension 2^n is the one defined in terms of a negative inner product :

$$\langle x, y \rangle = - \sum_{i=1}^n x_i y_i.$$

Thus $\|x\|^2 = -x^2$. Under this structure we have :

$$e_{ij} + e_{ji} = -2\delta_{ij}$$

where δ_{ij} is the Kronecker delta. This algebra is called a Clifford algebra and is denoted by Cl_n . Every element in this algebra is represented by

$$x = \sum_A e_A x_A$$

where $e_A = e_{i_1 i_2 \dots i_r}$ for $A = \{i_1 < i_2 < \dots < i_n\} \subseteq \{1, 2, 3, \dots, n\}$ and $x_A \in \mathbb{R}$.

Thus by identifying the element $x = (x_1, x_2, \dots, x_n)$ of \mathbb{R}^n with $\sum_{i=1}^n e_i x_i \in Cl_n$, we imbed the Euclidean space

$$\mathbb{R}^n \hookrightarrow Cl_n.$$

For $x, y \in Cl_n$, their Clifford product xy is written as a sum of their inner product and their anti-symmetric Grassmanian product, as :

$$xy = x \cdot y + x \wedge y.$$

In particular, for $x, y \in \mathbb{R}^n$, we have:

$$\begin{aligned} xy &= \left(\sum_{i=1}^n e_i x_i \right) \left(\sum_{j=1}^n e_j x_j \right) \\ &= \sum_{i,j=1}^n e_{ij} x_i x_j \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=j=1}^n e_{ii}x_iy_i + \sum_{i \neq j}^n e_{ij}x_iy_j \\
&= \underbrace{-\sum_{i=1}^n x_iy_i}_{x \cdot y} - \underbrace{\sum_{i < j}^n e_{ij}(x_iy_j - x_jy_i)}_{x \wedge y}
\end{aligned}$$

Every non zero element of \mathbb{R}^n is invertible : for $x \in \mathbb{R}^{n*}$, where $*$ indicates the tossing out of zero,

$$x^{-1} = \frac{-x}{\|x\|^2}.$$

Also for every element

$$x = \sum_{A \subseteq \{1 < 2 < \dots < n\}} e_A x_A$$

of $Cl_n(\mathbb{R})$, we define the Clifford conjugate \bar{x} of x by

$$\bar{x} := \sum_{A \subseteq \{1 < 2 < \dots < n\}} \bar{e}_A x_A$$

where, for

$$e_A = e_{i_1} \dots e_{i_k}$$

$$\bar{e}_A = \bar{e}_{i_k} \dots \bar{e}_{i_1}, \bar{e}_j = -e_j, j = 1, \dots, n, \bar{e}_0 = e_0$$

and therefore, we have a Clifford norm given by

$$\|x\|_{Cl} = [x\bar{x}]_0.$$

Thus, the unit sphere S^{n-1} in \mathbb{R}^n is described as :

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_{Cl_n} = 1\}.$$

Consider a c^1 - domain $\Omega \subseteq S^{n-1}$, a function $f : \Omega \rightarrow Cl_n$ has a representation given by :

$$f(x) = \sum_{A \subseteq \{1 < 2 < \dots < n\}} e_A f_A(x)$$

where $f_A : \Omega \rightarrow \mathbb{R}$. In this regard, a Clifford valued function over a domain is said to be C^k if each component real valued function f_A is C^k , and we say such a function belongs to a Sobolev space $W^{p,k}(\Omega, Cl_n)$ if each component function $f_A \in W^{p,k}(\Omega, Cl_n)$.

Let $f \in c^1(\Omega, Cl_n) \cap c(\bar{\Omega}, Cl_n)$, $\alpha \in \mathbb{C}$ and $\omega \in S^{n-1}$. Then

Definition 1. We define a spherical Dirac operator by

$$\Gamma_\alpha := \Gamma_\omega + \alpha$$

where

$$\begin{aligned} \Gamma_\omega &= -\omega \wedge D_\omega \\ &= -\sum_{i < j}^n e_{ij} \left(\omega_i \frac{\partial}{\partial \omega_j} - \omega_j \frac{\partial}{\partial \omega_i} \right) \end{aligned}$$

and $D_\omega = \sum_{i=1}^n e_i \frac{\partial}{\partial \omega_i}$ is the usual Dirac operator.

Definition 2. A function $f \in C^1(\Omega \rightarrow Cl_n)$ is called a spherical left monogenic function of order α if

$$\Gamma_\alpha f(x) = 0, \forall x \in \Omega$$

and is a spherical right monogenic function of order α if

$$f(x)\Gamma_\alpha = 0, \forall x \in \Omega.$$

Consider the generalized Gegenbauer function of degree α and of order λ :

$$C_\alpha^\lambda(z) = \frac{\Gamma(\alpha + 2\lambda)}{\Gamma(\alpha + 1)\Gamma(2\lambda)} F\left(-\alpha, \alpha + 2\lambda; \lambda + \frac{1}{2}; \frac{1}{2}(1 - z)\right)$$

where $F(a, b; c; d)$ is a hypergeometric function given by

$$F(a, b; c; d) := \sum_{k=1}^{\infty} \frac{(a)_k (b)_k d^k}{(c)_k k!}$$

for $|d| \leq 1$ with

$$(x)_k := \frac{\Gamma(x + k)}{\Gamma(x)}$$

which is simplified to: $\prod_{i=1}^k (x + i - 1)$

Proposition 1. The Gegenbauer function with degree α and order λ can be re-written as:

$$C_\alpha^\lambda(z) = \frac{\Gamma(\alpha + 2\lambda)}{\Gamma(\alpha + 1)\Gamma(2\lambda)} \sum_{k=1}^{\infty} \left(\prod_{i=1}^k \left(\frac{-\alpha(\alpha + 2\lambda) + (i-1)(2\lambda) + (i-1)^2}{\lambda - \frac{1}{2} + i} \right) \right) \frac{(1-z)^k}{k!2^k}$$

Proof. The proof follows from the simplification of the right side of the hypergeometric function

$$F(a, b; c; d) := \sum_{k=1}^{\infty} \frac{(a)_k (b)_k d^k}{(c)_k k!}$$

to the sum

$$\sum_{k=1}^{\infty} \left(\prod_{i=1}^k \left(\frac{ab + (i-1)(a+b) + (i-1)^2}{c+i-1} \right) \right) \frac{d^k}{k!}$$

and therefore the hypergeometric function with particular inputs $F\left(-\alpha, \alpha + 2\lambda; \lambda + \frac{1}{2}; \frac{1}{2}(1-z)\right)$ is simplified to

$$\sum_{k=1}^{\infty} \left(\prod_{i=1}^k \left(\frac{-\alpha(\alpha + 2\lambda) + (i-1)(2\lambda) + (i-1)^2}{\lambda - \frac{1}{2} + i} \right) \right) \frac{(1-z)^k}{k!2^k}.$$

Hence the Gegenbauer function is given by

$$C_{\alpha}^{\lambda}(z) = \frac{\Gamma(\alpha + 2\lambda)}{\Gamma(\alpha + 1)\Gamma(2\lambda)} \sum_{k=1}^{\infty} \left(\prod_{i=1}^k \left(\frac{-\alpha(\alpha + 2\lambda) + (i-1)(2\lambda) + (i-1)^2}{\lambda - \frac{1}{2} + i} \right) \right) \frac{(1-z)^k}{k!2^k}.$$

□

From the above Gegenbauer function, a function is constructed to be a fundamental solution (or Cauchy kernel) for the spherical Dirac operator $\Gamma_{\alpha} := \Gamma_{\omega} + \alpha$ as

$$\Psi_{\alpha}^n(\omega, v) = \frac{\pi}{\sigma_{n-1} \sin \pi \alpha} \left(C_{\alpha}^{\frac{n+1}{2}}(\omega.v) - \omega v C_{\alpha-1}^{\frac{n+1}{2}}(\omega.v) \right)$$

See [12],[10] for details.

Proposition 2. *The fundamental solution $\Psi_{\alpha}^n(\omega, v)$ to the spherical Dirac operator can be written as:*

$$\Psi_{\alpha}^n(\omega, v) = \frac{\pi}{\sigma_{n-1} \sin \pi \alpha} \left(\frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)\Gamma(n+1)} \sum_{k=1}^{\infty} \left[\left(\prod_{i=1}^k \left(\frac{-\alpha(\alpha+n+1) + (i-1)(n+1) + (i-1)^2}{\frac{n}{2} + i} \right) \right) - \omega v \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} \sum_{k=1}^{\infty} \left(\prod_{i=1}^k \left(\frac{(-\alpha+1)(\alpha+n) + (i-1)(n+1) + (i-1)^2}{\frac{n}{2} + i} \right) \right) \right] \frac{(1-\omega.v)^k}{k!2^k} \right).$$

Then using this as a Cauchy kernel, we define the following integral transforms over function spaces which are $C^{1,\alpha}$, for $0 \leq \alpha < 1$ or over Sobolev spaces $W^{p,k}(\Omega, Cl_n)$ for $1 < p < \infty$.

Let Ω be a bounded smooth domain in S^{n-1} and $f \in C^1(\Omega, Cl_n)$, then we define, the Teodorescu transform as:

$$T_{\Omega}(f)(v) = \int_{\Omega} \Psi_{\alpha}^n(\omega, v) f(\omega) d\omega, \text{ for } v \in S^{n-1}$$

which is the right inverse of the spherical Dirac operator Γ_{α} .

Also we have a non-singular boundary integral operator given by

$$F_{\partial\Omega}f(v) = \int_{\partial\Omega} \Psi_{\alpha}^n(\omega, v) n(v) f(v) d\partial\Omega_{\omega}, \text{ for } v \notin \partial\Omega$$

An other boundary integral is the singular integral given by

$$\tilde{F}_{\partial\Omega}f(v) = 2 \int_{\partial\Omega} \Psi_{\alpha}^n(\omega, v) n(v) f(v) d\partial\Omega_{\omega}, \text{ for } v \in \partial\Omega$$

The last integral is seen in terms of the Cauchy principal value and is good for computing non tangential limits of integrable functions on the boundary and also for Plemelji formulae.

By arguments of continuity and denseness, the integral transforms can also be extended over Sobolev spaces.

Also for $p \in (1, \infty)$ and $k = 0, 1, 2, \dots$, the following mapping properties hold:

$$T_{\Omega} : W^{p,k}(\Omega, Cl_n) \rightarrow W^{p,k+1}(\Omega, Cl_n)$$

and

$$F_{\partial\Omega} : W^{p,k-\frac{1}{p}}(\partial\Omega, Cl_n) \rightarrow W^{p,k}(\Omega, Cl_n)$$

Note that the functions in $W^{p,k-\frac{1}{p}}(\partial\Omega, Cl_n)$ are fractionally (or rationally) smooth and the $F_{\partial\Omega}$ is an operator which increases the smoothness of a function in the Slobedekij space $W^{p,k-\frac{1}{p}}(\partial\Omega, Cl_n)$ by $\frac{1}{p}$, and hence it maps functions from Slobedekij spaces to Sobolev spaces.

That is, the boundary transform $F_{\partial\Omega}$ retrieves regularity(smoothness) exponents of functions in $W^{p,k}(\Omega)$ which were lost by the trace operator as:

$$tr_{\partial\Omega} : W^{p,k}(\Omega, Cl_n) \rightarrow W^{p,k-\frac{1}{p}}(\partial\Omega, Cl_n) \text{ and } F_{\partial\Omega}f = F_{\partial\Omega}(tr_{\partial\Omega}f)$$

In general, the function spaces $W^{p,\gamma}(\partial\Omega, Cl_n)$, for γ a fraction are called Slobedekij spaces with the following definition :

Definition 3. $f \in W^{p,\gamma}(\partial\Omega, Cl_n)$ if $(1 + |\xi|^{-\gamma}) \hat{f} \in L^p(\partial\Omega, Cl_n)$

where \hat{f} is the Fourier transform of f and the norm is therefore given by

$$\|f\|_{W^{p,\gamma}(\partial\Omega, Cl_n)} := \left(\int_{\partial\Omega} (1 + |\xi|^\gamma)^p |\hat{f}|^p d\partial\Omega \right)^{\frac{1}{p}}$$

and these function spaces are used as spaces of symbols of pseudo-differential operators, in which, singular integral operators are special types of pseudo-differential operators.

Symbols are strong tools to study boundedness of pseudo-differential operators, where the symbol of a singular integral operator is bounded if and only if the operator is bounded, see [11]. In particular, it is indicated in [4], [11] that the π -operator is bounded by showing its symbol is bounded.

Proposition 3. *Let $f \in BC^1(\Omega \rightarrow Cl_n)$, with a bounded derivative. Then*

$$\Gamma_\alpha T_\Omega f = f.$$

That is T_Ω is a right inverse of Γ_α .

Theorem 1. (Borel-Pompeiu) *For $f \in C^1(\Omega \rightarrow Cl_n)$, we have*

$$\chi_\Omega f = F_{\partial\Omega} f + T_\Omega \Gamma_\alpha f$$

where χ_Ω is the usual characteristic function of the domain Ω .

Corollary 1. (Cauchy Integral Formula for Spherical Monogenics)

$$f \in \ker \Gamma_\alpha \Leftrightarrow f(v) = F_{\partial\Omega} f(v)$$

Corollary 2. *From the Borel-Pompeiu and the CIFs, a traceless γ -regular function is a null function over Ω .*

3. Fundamental Results on the Spherical Dirac Operator

In this section, we present fundamental results on Γ_α , solve boundary value problems over domains in the unit sphere like cases of domains in Euclidean spaces, using the algebraic and analytic tools presented in the preliminary.

Proposition 4. *Let $g \in W^{2,1}(\Omega, Cl_n)$, $h \in W^{2,\frac{1}{2}}(\partial\Omega, Cl_n)$. Then the inhomogeneous BVP:*

$$\begin{cases} \Gamma_\alpha f = g \text{ on } \Omega \\ \text{tr} f = h, \text{ on } \partial\Omega \end{cases}$$

has a unique solution $f \in W^{2,2}(\Omega, Cl_n)$ given by

$$f = F_{\partial\Omega} h + T_\Omega g.$$

which is almost a C^2 -function for no $\mathbb{R}^n \supseteq \Omega$ but is almost a C^1 -function in \mathbb{R}^1

Proof. The unique solution of the BVP is obtained using the Borel-Pompeiu formula which is given by

$$f = F_{\partial\Omega}h + T_{\Omega}g.$$

□

Corollary 3. *The analytic solution f of the BVP given above is almost a C^1 -function in \mathbb{R}^1 but almost a C^2 -function for no \mathbb{R}^n .*

Proof. The solution f given above is a function in the Sobolev space $W^{2,2}(\Omega, Cl_n)$ and is almost a C^k -function over Ω contained in \mathbb{R}^n , if

$$2 > \frac{n}{2} + k$$

where $k \in \mathbb{N}$.

But the last inequality holds only when $k = 1$ and $n = 1$ and that prove the result. □

Proposition 5. *(Representation of a Function with Compact Support)*

Let $f \in C_c^1(\Omega \rightarrow Cl_n)$. Then f has a representation given by

$$f(v) = T_{\Omega}(\Gamma_{\alpha}f)(v)$$

for $v \in S^{n-1}$.

Proof. Let f be a C^1 -function with compact support over $\Omega \subseteq S^{n-1}$. Then from Borel-Pompeiu formula we have the required result, since the boundary integral is zero.

We see here that T_{Ω} is both right and left inverse of the spherical Dirac operator Γ_{α} . □

Proposition 6. *(Representation of a Global Function)*

If Ω is a global domain in the unit sphere, then every function f in $W^{2,2}(\Omega, Cl_n)$ (or in $C^1(\Omega, Cl_n)$) can be represented over Ω by

$$f(v) = \int_{\Omega} \Psi_{\alpha}^n(w-v) \Gamma_{\alpha}(w) f(w) d\Omega_w.$$

Proof. Ω is a global domain in the unit sphere means that Ω is the whole sphere. Thus as the sphere is a boundary hypersurface,

its boundary is empty set (we recall from differential topology that $\partial\partial = \emptyset$). Therefore the ∂ -integral of f :

$$\begin{aligned} \int_{\partial\Omega} \Psi_{\alpha}^n(w-v) n(w) f(w) d\Omega_w &= \int_{\partial\Omega=\partial\partial(\bullet)} \Psi_{\alpha}^n(w-v) n(w) f(w) d\Omega_w \\ &= \int_{\emptyset} \Psi_{\alpha}^n(w-v) n(w) f(w) d\Omega_w \\ &= 0 \end{aligned}$$

and therefore from Borel-Pompeiu formula, we have the result. □

The Lebesgue space $L^2(\Omega, Cl_n)$ with a Clifford valued inner product given by

$$(3.1) \quad \langle f, g \rangle_{\Omega} = \int_{\Omega} \bar{f}g d\Omega$$

$f, g \in L^2(\Omega, Cl_n)$ is a Hilbert space and therefore has an orthogonal relationship given by:

Proposition 7. *In the Hilbert space $L^2(\Omega, Cl_n)$, with respect to the inner product (3.1) the orthogonal space $(B_{\alpha}^2(\Omega, Cl_n))^{\perp}$ of the generalized Bergman space $B_{\alpha}^2(\Omega, Cl_n)$ is given by :*

$$(B_{\alpha}^2(\Omega, Cl_n))^{\perp} = \bar{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right)$$

where the Bergman space $B_{\alpha}^2(\Omega, Cl_n)$ is the set of all Clifford valued square integrable functions which are annihilated by the spherical Dirac operator Γ_{α} over Ω .

Proof. First lets prove that $B_{\alpha}^2(\Omega, Cl_n) \cap \bar{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right)$ is $\{0\}$, the singleton with only the zero function as the element.

Indeed, for $f \in B_{\alpha}^2(\Omega, Cl_n) \cap \bar{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right)$, we have $\Gamma_{\alpha}f = 0$ on Ω and $f = \bar{\Gamma}_{\alpha}g$, for $g \in W_0^{2,1}(\Omega, Cl_n)$.

Then $\Gamma_{\alpha}f = \Delta_{\alpha,0}g = 0 \Rightarrow g \equiv 0$ on Ω . Therefore $f \equiv 0$ on Ω .

Also for $f \in L^2(\Omega, Cl_n)$ we have $f = Pf + Qf$ with $Pf = f - \Gamma_{\alpha} \left(\Delta_{\alpha,0}^{-1} \Gamma_{\alpha}f \right)$ and $Qf = \Gamma_{\alpha} \left(\Delta_{\alpha,0}^{-1} \Gamma_{\alpha}f \right)$ with $Pf \in B_{\alpha}^2(\Omega, Cl_n)$ and $Qf \in \bar{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right)$, where P is the Bergman projection and Q is its orthogonal complement. \square

As usual, the two orthogonal projections we use in the proof of the above orthogonality relations are

$$P : L^2(\Omega, Cl_n) \rightarrow B_{\alpha}^2(\Omega, Cl_n)$$

which is the Bergman projection and

$$Q : L^2(\Omega, Cl_n) \rightarrow \bar{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right)$$

is its orthogonal complement with

$$Q = I - P, \text{ and}$$

$$PQ = 0 = QP, P^2 = P, Q^2 = Q.$$

Proposition 8. For $\phi \in B_\alpha^2(\Omega, Cl_n)$ and $\psi \in (B_\alpha^2(\Omega, Cl_n))^\perp$, the squared norm defined by $||| \bullet ||| := \| \bullet \|_{L^2(\Omega, Cl_n)}^2$ do satisfy the following equalities: $\forall n \in \mathbb{N}$,

$$\begin{aligned} (a) \quad & \| \phi + \psi \|_{L^2(\Omega, Cl_n)}^n = \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^{\frac{n}{2}} \\ (b) \quad & ||| \phi + \psi |||^n = (||| \phi ||| + ||| \psi |||)^n \end{aligned}$$

Proof. Here the proof can be done using induction on n :

Since $\phi \in \langle \psi \rangle^\perp$, the orthogonal space of the space $\langle \psi \rangle$ spanned by ψ , we have $\int_\Omega \overline{\phi} \psi d\Omega = 0 = \int_\Omega \overline{\psi} \phi d\Omega$ which implies,

$$\begin{aligned} \| \phi + \psi \|_{L^2(\Omega, Cl_n)}^2 &= \langle \phi + \psi, \phi + \psi \rangle_\Omega \\ &= \int_\Omega \overline{(\phi + \psi)} (\phi + \psi) d\Omega = \int_\Omega \overline{\phi} \phi d\Omega + \int_\Omega \overline{\psi} \psi d\Omega \\ &= \| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \end{aligned}$$

That is

$$\| \phi + \psi \|_{L^2(\Omega, Cl_n)} = \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^{\frac{1}{2}}$$

Therefore the statement is valid for $n = 1$. We assume it is true for k , that is

$$\| \phi + \psi \|_{L^2(\Omega, Cl_n)}^k = \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^{\frac{k}{2}}$$

Then

$$\begin{aligned} \| \phi + \psi \|_{L^2(\Omega, Cl_n)}^{k+1} &= \| \phi + \psi \|_{L^2(\Omega, Cl_n)}^k \| \phi + \psi \|_{L^2(\Omega, Cl_n)} \\ &= \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^{\frac{k}{2}} \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^{\frac{1}{2}} \\ &= \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^{\frac{k+1}{2}} \end{aligned}$$

which shows the validity of the statement for $k + 1$ and that proves the statement for $\forall n \in \mathbb{N}$, and that proves (a).

(b) follows easily:

$$\begin{aligned} | \quad & ||| \phi + \psi |||^n = \left(\| \phi + \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^n \\ &= \left(\| \phi \|_{L^2(\Omega, Cl_n)}^2 + \| \psi \|_{L^2(\Omega, Cl_n)}^2 \right)^n \\ &= (||| \phi ||| + ||| \psi |||)^n \end{aligned}$$

□

In [9], the authors have a decomposition result for Sobolev spaces

$$W^{p,k-1}(\Omega, Cl_n) = B^{p,k}(\Omega, Cl_n) \dot{+} D^k \left(\overset{0}{W}{}^{p,2k-1}(\Omega, Cl_n) \right)$$

where $\dot{+}$ is a direct sum and when $p = 2$ it is an orthogonal sum with respect to the inner product (3.1), with corresponding orthogonal projections

$$P^{(k)} : W^{2,k-1}(\Omega, Cl_n) \rightarrow B^{2,k}(\Omega, Cl_n)$$

and

$$Q^{(k)} : W^{2,k-1}(\Omega, Cl_n) \rightarrow D^k \left(\overset{0}{W}{}^{2,2k-1}(\Omega, Cl_n) \right)$$

with $Q^{(k)} = I - P^{(k)}$ such that

$$P^{(k)}Q^{(k)} = Q^{(k)}P^{(k)} = 0, \left(P^{(k)}\right)^2 = P^{(k)}, \left(Q^{(k)}\right)^2 = Q^{(k)}$$

and $D^k = \left(\sum_{i=0}^n e_i \frac{\partial}{\partial x_i}\right)^k$, is the k^{th} iterate of the Dirac operator.

Proposition 9. For $f \in L^2(\Omega, Cl_n)$, and P , the Bergman projection, we have

$$\langle Pf, f \rangle_{\Omega} = \langle Pf, Pf \rangle_{\Omega}$$

Proof. Let $f \in L^2(\Omega, Cl_n)$. Then

$$f = Pf + Qf$$

with $Pf \in B_{\alpha}^2(\Omega, Cl_n)$ and $Qf \in \bar{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right)$.

Therefore,

$$\langle Pf, Qf \rangle_{\Omega} = \langle Pf, Pf \rangle_{\Omega} = \int_{\Omega} \overline{Pf} Qf d\Omega = 0$$

which implies

$$\int_{\Omega} \overline{Pf} (I - Q) f d\Omega = \int_{\Omega} (\overline{Pf} f - \overline{Pf} Qf) d\Omega = 0.$$

That is,

$$\int_{\Omega} \overline{Pf} f d\Omega = \int_{\Omega} \overline{Pf} Qf d\Omega.$$

Therefore,

$$\langle Pf, f \rangle_{\Omega} = \langle Pf, Pf \rangle_{\Omega}.$$

□

Proposition 10. For $f \in L^2(\Omega, Cl_n)$, $\exists g \in W_0^{2,1}(\Omega, Cl_n)$ such that

$$\Gamma_{\alpha} f = \Gamma_{\alpha} \bar{\Gamma}_{\alpha} g.$$

Proof. Let $f \in L^2(\Omega, Cl_n)$. Then

$$f = Pf + Qf$$

with $Pf \in B^2(\Omega, Cl_n)$ and $Qf \in \bar{\Gamma}_\alpha(W_0^{2,1}(\Omega, Cl_n))$.

Therefore there exists $g \in W_0^{2,1}(\Omega, Cl_n)$ such that $Qf = \bar{\Gamma}_\alpha g$. Then applying Γ_α on both sides of the equation

$$f = Pf + Qf$$

we have the required result. \square

Remark 1. *In the case where the Laplacian is factored as*

$$\Delta = D\bar{D} = \bar{D}D$$

we could have that $Df = \Delta g$, but in the case of spherical Laplacian the factorization is a bit different.

For Γ_α the spherical Dirac operator, the Spherical Laplacian is factored as

$$\Delta_\alpha = \Gamma_\alpha \Gamma_\beta = \Gamma_\beta \Gamma_\alpha$$

where $\alpha + \beta = -n + 1$.

4. Results on the Spherical $\pi_{\alpha, S^{n-1}}$ -Operator

As is done in the case of defining the π -operator over general domains in Euclidean spaces, we define $\pi_{\alpha, S^{n-1}}$ over domains in the unit sphere as follows.

Definition 4. *For $f \in C^1(\Omega \rightarrow Cl_n)$, define*

$$\pi_{\alpha, S^{n-1}}(f) := \bar{\Gamma}_\alpha T_\Omega(f).$$

In the scale of Sobolev spaces, $\pi_{\alpha, S^{n-1}}$ is an operator from $W^{p,k}(\Omega, Cl_n) \rightarrow W^{p,k}(\Omega, Cl_n)$, for $1 < p < \infty, k = 0, 1, 2, \dots$

Over domains Ω in Euclidean spaces \mathbb{R}^n (or \mathbb{C}^n), this operator has the following integral representation:

For $n = 1$:

$$\pi_\Omega f(w) = \int_\Omega \Psi(z-w) f(z) dz$$

where

$$\Psi(z-w) = \frac{-1}{\pi(z-w)^2}$$

and for $n > 1$, we have a representation given by :

$$\pi_\Omega f(x) = \int_\Omega -\frac{n + (n+2) \frac{\overline{(y-x)}^2}{|y-x|^2}}{\omega |y-x|^{n+2}} f(y) d\Omega_y + \frac{-n}{n+2} f(x)$$

which in both cases the π_{Ω} -operator is a hyper singular integral operator of C-Z type.

Note also that from the above general formula of the π_{Ω} - operator, taking $n = 0$, we have the well known π_{Ω} -operator (given above) in the usual 1D complex space $\mathbb{C}^1 \simeq Cl_{0,1}$, where $\sqrt{-1} = i \sim e_1$.

Also in a generalized setting, Dejenie A. Lakew and John Ryan in [8] study the π -operator over real, compact $(n + 1)$ - manifolds in \mathbb{C}^{n+1} which are called Domain Manifolds and produced an integral representation of π over such manifolds given by:

$$\pi_{\Omega}f(w) = \int_{\Omega} \overline{D}_{\gamma} \Psi^{\Gamma}(w - v) f(v) d\Omega_v + \frac{-n + 1}{n + 1} f(w)$$

where D_{γ} is a nonhomogeneous Dirac like operator defined by $D_{\gamma} = \sum_{j=0}^n e_j \left(\frac{\partial}{\partial x_j} - \gamma_j \right)$, Ψ^{Γ} is a fundamental solution for D_{γ} and Ω is a domain manifold in \mathbb{C}^{n+1} .

Definition 5. We define the Clifford conjugate of $\pi_{\alpha, S^{n-1}}$ as

$$\overline{\pi}_{\alpha, S^{n-1}} := \Gamma_{\alpha} \overline{T}_{\Omega}$$

where

$$\overline{T}_{\Omega}(f)(x) = \int_{\Omega} \overline{\Psi}_{\alpha}^n(y, x) f(y) d\Omega_y.$$

Proposition 11. (Classical Analogous Results:[4],[5])

On the Sobolev space $W^{p,k}(\Omega, Cl_n)$, where $1 < p < \infty$, $k = 0, 1, 2, \dots$, we have :

- (1) $\Gamma_{\alpha} \pi_{\alpha, S^{n-1}} = \overline{\Gamma}_{\alpha}$
- (2) $\pi_{\alpha, S^{n-1}} \Gamma_{\alpha} = \overline{\Gamma}_{\alpha} (I - F_{\partial\Omega})$
- (3) $F_{\partial\Omega} \pi_{\alpha, S^{n-1}} = \pi_{\alpha, S^{n-1}} - T_{\Omega} \overline{\Gamma}_{\alpha}$
- (4) $\Gamma_{\alpha} \pi_{\alpha, S^{n-1}} - \pi_{\alpha, S^{n-1}} = \overline{\Gamma}_{\alpha} F_{\partial\Omega}$

Corollary 4. From the above proposition we see that

$$\pi_{\alpha, S^{n-1}} : \overline{B}_{\alpha}^2(\Omega, Cl_n) \rightarrow B_{\alpha}^2(\Omega, Cl_n)$$

where

$$\overline{B}_{\alpha}^2(\Omega, Cl_n) = L^2(\Omega, Cl_n) \cap \ker \overline{\Gamma}_{\alpha}(\Omega, Cl_n) \quad \text{and} \quad B_{\alpha}^2(\Omega, Cl_n)$$

is the Bergman space $L^2(\Omega, Cl_n) \cap \ker \Gamma_{\alpha}(\Omega, Cl_n)$

Corollary 5. Also we have

$$\pi_{\alpha, S^{n-1}} : \Gamma_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right) \rightarrow \overline{\Gamma}_{\alpha} \left(W_0^{2,1}(\Omega, Cl_n) \right).$$

Proposition 12. Let $g \in W_0^{p,k}(\Omega, Cl_n)$, $1 < p < \infty$, $k = 0, 1, 2, \dots$. Then

- (1) $\pi_{\alpha, S^{n-1}} \Gamma_{\alpha} g = \overline{\Gamma}_{\alpha} g$

$$(2) \quad \Gamma_\alpha \pi_{\alpha, S^{n-1}} = \pi_{\alpha, S^{n-1}}$$

$$(3) \quad \pi_{\alpha, S^{n-1}} \Gamma_\alpha g = \Gamma_\alpha \pi_{\alpha, S^{n-1}} g = \pi_{\alpha, S^{n-1}} g$$

Proof. If $g \in W^{p,k}(\Omega, Cl_n)$ is compactly supported over Ω , then the boundary integral of g is zero. That is $F_{\partial\Omega} g = 0$, and therefore, from

$$\pi_{\alpha, S^{n-1}} \Gamma_\alpha = \bar{\Gamma}_\alpha (I - F_{\partial\Omega})$$

we have

$$\pi_{\alpha, S^{n-1}} \Gamma_\alpha = \bar{\Gamma}_\alpha$$

since the ∂ -integral is zero, which proves (1)

Also from

$$\Gamma_\alpha \pi_{\alpha, S^{n-1}} g - \pi_{\alpha, S^{n-1}} g = \bar{\Gamma}_\alpha F_{\partial\Omega} (tr_{\partial\Omega} g)$$

as g is compactly supported over the domain, its boundary integral over the domain is zero, i.e. $F_{\partial\Omega} (tr_{\partial\Omega} g) = 0$.

Thus we have

$$\Gamma_\alpha \pi_{\alpha, S^{n-1}} g - \pi_{\alpha, S^{n-1}} g = \bar{\Gamma}_\alpha F_{\partial\Omega} (tr_{\partial\Omega} g) = 0$$

which implies

$$\Gamma_\alpha \pi_{\alpha, S^{n-1}} g = \pi_{\alpha, S^{n-1}} g$$

which proves (2).

Finally,

$$\Gamma_\alpha \pi_{\alpha, S^{n-1}} g = \bar{\Gamma}_\alpha g$$

on the Sobolev space $W^{p,k}(\Omega, Cl_n)$ and in particular when $tr_{\partial\Omega} g = 0$, we have

$$\pi_{\alpha, S^{n-1}} \Gamma_\alpha g = \bar{\Gamma}_\alpha g$$

adjoining this with result (2) we prove (3). \square

Remark 2. *On the space of functions with compact support, the action of the spherical Dirac operator from the right and from the left on $\pi_{\alpha, S^{n-1}}$ is irrelevant.*

Corollary 6. *If f is a smooth Clifford valued function which has a compact support over Ω , then $\pi_{\alpha, S^{n-1}}$ and Γ_α commute at f and furthermore, their product at such a function is the Clifford conjugate $\bar{\Gamma}_\alpha$ of Γ_α .*

Proposition 13. *In the Sobolev space $W^{2,1}(\Omega, Cl_n)$,*

$$f \in \ker \bar{\Gamma}_\alpha \Rightarrow \pi_{\alpha, S^{n-1}} f \in \ker \Gamma_\alpha$$

That is

$$\pi_{\alpha, S^{n-1}} : W^{2,1}(\Omega) \cap \ker \bar{\Gamma}_\alpha \rightarrow W^{2,1}(\Omega) \cap \ker \Gamma_\alpha.$$

Proof. Let $f \in W^{2,1}(\Omega, Cl_n) \cap \ker \bar{\Gamma}_\alpha$. Then $\pi_{\alpha, S^{n-1}} f \in W^{2,1}(\Omega, Cl_n)$ as $\pi_{\alpha, S^{n-1}}$ preserves regularity, and $\bar{\Gamma}_\alpha f = 0$. But from the relation

$$\bar{\Gamma}_\alpha = \Gamma_\alpha \pi_{\alpha, S^{n-1}}$$

we get

$$\Gamma_\alpha \pi_{\alpha, S^{n-1}} f = 0$$

which is the required result. From this result, one can see that the π_α operator preserves monogenicity or hypercomplex regularity of functions in the sense :

$$\pi_{\alpha, S^{n-1}} : \bar{B}_\alpha^2(\Omega, Cl_n) \rightarrow B_\alpha^2(\Omega, Cl_n)$$

where

$$\bar{B}_\alpha^2(\Omega, Cl_n) = L^2(\Omega, Cl_n) \cap \ker \bar{\Gamma}_\alpha \text{ and } B_\alpha^2(\Omega, Cl_n)$$

is the Bergman space mentioned above. \square

Proposition 14. *Let*

$$\Lambda_{0,j} := e_0 e_j \det \begin{pmatrix} \omega_0 & \omega_j \\ \frac{\partial}{\partial \omega_0} & \frac{\partial}{\partial \omega_j} \end{pmatrix}, j = 1, \dots, n.$$

If $\pi_{\alpha, S^{n-1}}$ fixes $f \in L^2(\Omega, Cl_n)$, then f satisfies the equation

$$f = \left(\sum_{0 < j} \Lambda_{0,j} + \alpha \right) T_\Omega f.$$

Proof. First we rewrite the conjugate of the spherical Dirac operator $\bar{\Gamma}_\alpha$ as

$$\bar{\Gamma}_\alpha = 2 \left(\sum_{0 < j} e_0 e_j \underbrace{\left(\omega_0 \frac{\partial}{\partial \omega_j} - \omega_j \frac{\partial}{\partial \omega_0} \right)}_{\Lambda_{0,j}} + \alpha \right) - \Gamma_\alpha.$$

Then

$$\pi_{\alpha, S^{n-1}} f = f$$

implies

$$\bar{\Gamma}_\alpha T_\Omega f = f.$$

Using the expression for $\bar{\Gamma}_\alpha$ in terms of Γ_α given above we get the desired result. \square

Remark 3. *From the above result and the orthogonal decomposition of the Hilbert space, we can see that the spherical $\pi_{\alpha, S^{n-1}}$ has the mapping property:*

$$\pi_{\alpha, S^{n-1}} : \bar{\Gamma}_\alpha \left(W_0^{2,1}(\Omega, Cl_n) \right) \rightarrow \Gamma_\alpha \left(W_0^{2,1}(\Omega, Cl_n) \right)$$

pictorially, we describe the above mapping properties as (for further studies, see [4],[5]):

$$L^2(\Omega, Cl_n) = \left. \begin{array}{c} \underbrace{B_\alpha^2(\Omega, Cl_n)}_{\downarrow} \oplus \underbrace{\bar{\Gamma}_\alpha(W_0^{2,1}(\Omega, Cl_n))}_{\downarrow} \\ \bar{B}_\alpha^2(\Omega, Cl_n) \oplus \Gamma_\alpha(W_0^{2,1}(\Omega, Cl_n)) \end{array} \right\} \downarrow: \pi_{\alpha, S^{n-1}}$$

Proposition 15. *Let $p \in (1, \infty)$, $k \in \mathbb{Z} \cup \{0\}$ and $f \in W^{p,k}(\Omega, Cl_n)$. Then*

$$\bar{\pi}_{\alpha, S^{n-1}} \pi_{\alpha, S^{n-1}} f + \Gamma_\alpha \bar{F}_{\partial\Omega} T_\Omega f = f.$$

Corollary 7. *By taking the complexified Clifford conjugate of the above equation we get*

$$\pi_{\alpha, S^{n-1}} \bar{\pi}_{\alpha, S^{n-1}} f + \bar{\Gamma}_\alpha F_{\partial\Omega} \bar{T}_\Omega f = f$$

Corollary 8. *The π_{α, S^n} operator is left invertible on the space $\Gamma_\alpha(W_0^{2,1}(\Omega, Cl_n))$ with left inverse of $\bar{\pi}_{\alpha, S^{n-1}}$.*

Proof. Let $f \in \Gamma_\alpha(W_0^{2,1}(\Omega, Cl_n))$. Then there exists a function $g \in W_0^{2,1}(\Omega, Cl_n)$ such that $f = \Gamma_\alpha g$ with $tr_{\partial\Omega} g = 0$.

From the Borel-Pompeiu formula, we have

$$g = T_\Omega \Gamma_\alpha g = T_\Omega f$$

and this implies

$$tr_{\partial\Omega} T_\Omega f = 0$$

and therefore,

$$F_{\partial\Omega} T_\Omega f = F_{\partial\Omega} (tr_{\partial\Omega} T_\Omega f) = 0$$

which yields the result. \square

Remark 4. *A similar argument yields that $\pi_{\alpha, S^{n-1}}$ is right invertible on $\bar{\Gamma}_\alpha(W_0^{2,1}(\Omega, Cl_n))$ with right inverse of $\bar{\pi}_{\alpha, S^{n-1}}$.*

Denote by Ξ the overlap of the function spaces $\bar{\Gamma}_\alpha(W_0^{2,1}(\Omega, Cl_n)) \cap \Gamma_\alpha(W_0^{2,1}(\Omega, Cl_n))$.

Lemma 1. *Ξ is non empty.*

Proof. Here, to show that the set Ξ is non-empty, we need to consider the Sobolev space $(\Gamma_\alpha \bar{\Gamma}_\alpha)(W_0^{2,2}(\Omega, Cl_n))$ where $\bar{\Gamma}_\alpha$ is the Clifford conjugate of the spherical Dirac operator of order α . Let $f \in \Gamma_\alpha \bar{\Gamma}_\alpha(W_0^{2,2}(\Omega, Cl_n))$. Then $\exists g \in \Gamma_\alpha(W_0^{2,2}(\Omega, Cl_n))$ and $h \in \bar{\Gamma}_\alpha(W_0^{2,2}(\Omega, Cl_n)) \ni f = \Gamma_\alpha g = \bar{\Gamma}_\alpha h$.

That is $f \in \Xi$. In fact the argument shows that $\Gamma_\alpha \bar{\Gamma}_\alpha \left(W_0^{2,2}(\Omega, Cl_n) \right) \subseteq \Xi$. \square

Proposition 16. *On the space Ξ , $\pi_{\alpha, S^{n-1}}$ is invertible with inverse of $\bar{\pi}_{\alpha, S^{n-1}}$.*

When we consider global functions over the sphere we may have better results on invertibility of $\pi_{\alpha, S^{n-1}}$.

Proposition 17. *On the space $C_0^\infty(S^{n-1}, Cl_n)$, we have*

$$\bar{\pi}_{\alpha, S^{n-1}} \pi_{\alpha, S^{n-1}} = \pi_{\alpha, S^{n-1}} \bar{\pi}_{\alpha, S^{n-1}}$$

From denseness arguments, and boundedness of the $\pi_{\alpha, S^{n-1}}$ on $L^2(\Omega, Cl_n)$, the above result can be done over a larger domain as:

Corollary 9. *On the space $L^2(S^{n-1}, Cl_n)$, we have*

$$\bar{\pi}_{\alpha, S^{n-1}} \pi_{\alpha, S^{n-1}} = \pi_{\alpha, S^{n-1}} \bar{\pi}_{\alpha, S^{n-1}}$$

With respect to the Clifford valued inner product given by (3.1) on the Hilbert space $L^2(\Omega, Cl_n)$,

we take $-\bar{T}_\Omega$ as the adjoint T_Ω^* of T_Ω and $-\bar{\Gamma}_\alpha$ as Γ_α^* , adjoint of Γ_α .

Therefore for $f, g \in W_0^{2,k}(\Omega, Cl_n)$, we have

$$\begin{aligned} \langle \Gamma_\alpha f, g \rangle_\Omega &= \int_\Omega \overline{\Gamma_\alpha f} g d\Omega \\ &= \int_\Omega \bar{\Gamma}_\alpha \bar{f} g d\Omega \\ &= - \int_\Omega f \bar{\Gamma}_\alpha g d\Omega \\ &= - \langle f, \bar{\Gamma}_\alpha g \rangle_\Omega \end{aligned}$$

and

$$\begin{aligned} \langle T_\Omega f, g \rangle_\Omega &= \int_\Omega \overline{T_\Omega f} g d\Omega_v \\ &= \int_\Omega \bar{T}_\Omega \bar{f} g d\Omega_v \\ &= \int_\Omega \left(\int_\Omega \bar{\Psi}_\alpha^n(\omega, v) \bar{f}(\omega) d\Omega_\omega \right) g(v) d\Omega_v \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega \times \Omega} \overline{\Psi}_\alpha^n(\omega, v) \overline{f}(\omega) g(v) d\Omega_\omega d\Omega_v \\
&= - \int_{\Omega} \overline{f}(\omega) d\Omega_\omega \int_{\Omega} \overline{\Psi}_\alpha^n(\omega, v) g(v) d\Omega_v \\
&= - \int_{\Omega} \overline{f}(\omega) (\overline{T}_\Omega g(v) d\Omega_v) d\Omega_\omega \\
&= - \langle f, \overline{T}_\Omega g \rangle_\Omega.
\end{aligned}$$

Lemma 2. $\pi_\alpha^* = \overline{T}_\Omega \Gamma_\alpha$

Proof.

$$\begin{aligned}
\langle \pi_{\alpha, S^{n-1}} f, g \rangle &= \langle \overline{\Gamma}_\alpha T_\Omega f, g \rangle \\
&= - \langle T_\Omega f, \Gamma_\alpha g \rangle \\
&= \langle f, \overline{T}_\Omega \Gamma_\alpha g \rangle \\
&= \langle f, \pi_{\alpha, S^{n-1}}^* \rangle
\end{aligned}$$

From this we can see that the adjoint $\pi_{\alpha, S^{n-1}}^*$ of the $\pi_{\alpha, S^{n-1}}$ operator is $\overline{T}_\Omega \Gamma_\alpha$ \square

Proposition 18. *On $W_0^{2,k}(\Omega, Cl_n)$, $k = 0, 1, 2, \dots$, we have*

$$\pi_{\alpha, S^{n-1}}^* \pi_{\alpha, S^{n-1}} = I_{S^{n-1}}.$$

Proof. For $f \in W_0^{2,k}(\Omega, Cl_n)$,

$$\begin{aligned}
\pi_{\alpha, S^{n-1}}^* \pi_{\alpha, S^{n-1}} f &= \overline{T}_\Omega \Gamma_\alpha \pi_{\alpha, S^{n-1}} f \\
&= \overline{T}_\Omega \Gamma_\alpha \overline{\Gamma}_\alpha T_\Omega f \\
&= \overline{T}_\Omega \overline{\Gamma}_\alpha \Gamma_\alpha T_\Omega f \\
&= \overline{I}_\Omega I_\Omega f = f.
\end{aligned}$$

This is because for a function $f \in W^{2,k}(\Omega, Cl_n)$ whose trace is zero over the boundary, Γ_α is both the right and left inverse of the T_Ω operator. \square

Corollary 10. *On the Hilbert space $L^2(\Omega, Cl_n)$,*

$$\pi_{\alpha, S^{n-1}}^* = \overline{\pi}_{\alpha, S^{n-1}}.$$

From the above corollary we get that for $f, g \in L^2(\Omega, Cl_n)$,

$$\begin{aligned}
\langle \pi_{\alpha, S^{n-1}} f, \pi_{\alpha, S^{n-1}} g \rangle_\Omega &= \langle f, \pi_{\alpha, S^{n-1}}^* \pi_{\alpha, S^{n-1}} g \rangle_\Omega \\
&= \langle f, g \rangle_\Omega.
\end{aligned}$$

By taking $f = g$, we have an isometry property for the $\pi_{\alpha, S^{n-1}}$ operator over $L^2(\Omega, Cl_n)$:

Proposition 19. $\| \pi_{\alpha, S^{n-1}} f \| = \| f \|$ for $f \in W^{2,0}(\Omega, Cl_n)$, i.e., $\pi_{\alpha, S^{n-1}}$ is norm preserving over the Hilbert space.

Proof. For $f \in W^{2,0}(\Omega, Cl_n)$,

$$\begin{aligned} \| \pi_{\alpha, S^{n-1}} f \| &= \left[\langle \pi_{\alpha, S^{n-1}} f, \pi_{\alpha, S^{n-1}} f \rangle_{\Omega} \right]_0 \\ &= \left[\langle f, \pi_{\alpha, S^{n-1}}^* \pi_{\alpha, S^{n-1}} f \rangle_{\Omega} \right]_0 \\ &= \left[\langle f, f \rangle_{\Omega} \right]_0 = \| f \| \end{aligned}$$

□

In the following proposition we identify functions in the Hilbert space which are fixed by the spherical $\pi_{\alpha, S^{n-1}}$ operator?

Proposition 20. (Fixed Points of $\pi_{\alpha, S^{n-1}}$) Let $f \in L^2(\Omega, Cl_n)$. If

$$\pi_{\alpha, S^{n-1}} f = f$$

then

$$f = \left(\sum_{0 < j} e_0 e_j \left(\omega_0 \frac{\partial}{\partial \omega_j} - \omega_j \frac{\partial}{\partial \omega_0} \right) + \alpha \right) T_{\Omega} f.$$

Proof. First

$$\begin{aligned} \Gamma_{\alpha} &= \Gamma_{\omega} + \alpha \\ &= - \sum_{i < j} e_{ij} \left(\omega_i \frac{\partial}{\partial \omega_j} - \omega_j \frac{\partial}{\partial \omega_i} \right) + \alpha \end{aligned}$$

Then

$$\bar{\Gamma}_{\alpha} = -2 \left(\sum_{0 < j} e_0 e_j \left(\omega_0 \frac{\partial}{\partial \omega_j} - \omega_j \frac{\partial}{\partial \omega_0} \right) + \alpha \right) - \Gamma_{\alpha}.$$

This gives us

$$\pi_{\alpha, S^{n-1}} f = \left(-2 \left(\sum_{0 < j} e_0 e_j \left(\omega_0 \frac{\partial}{\partial \omega_j} - \omega_j \frac{\partial}{\partial \omega_0} \right) + \alpha \right) - \Gamma_{\alpha} \right) T_{\Omega} f$$

simplifying this and equating the result to f , we get the result. □

Remark 5. The $\pi_{\alpha, S^{n-1}}$ operator is a bounded, and isometric operator which preserves regularity over Sobolev spaces and yet different from the Identity operator .

Remark 6. For a global function $g \in L^2(S^{n-1}, Cl_n)$, or in $W_0^{2,k}(\Omega, Cl_n)$, the equation

$$\pi_{\alpha, S^{n-1}} f = g$$

has a solution in the respective space given by

$$f = \pi_{\alpha, S^{n-1}}^* g$$

where $\pi_{\alpha, S^{n-1}}^*$ is the adjoint of the $\pi_{\alpha, S^{n-1}}$ operator.

5. The Spherical Clifford Beltrami Equation

For measurable functions $f, q : \Omega \subseteq \mathbb{C}$ with $\|q\| < 1$, the classical Beltrami equation given by

$$f_z - qf_{\bar{z}} = 0$$

has been studied by many authors. The equation has also its version in higher dimensions in the real Clifford algebra $Cl_n(\mathbb{R})$ or in the complexified Clifford algebra $Cl_n(\mathbb{C})$ or over domain manifolds in \mathbb{C}^n .

In [8], the authors study the Beltrami equation over \mathbb{C}^{n+1} via real, compact, $(n+1)$ -manifolds in \mathbb{C}^{n+1} . This is possible by introducing an intrinsic Dirac operator specific to each domain manifold.

In this paper we extend our study of the π -operator over spherical domains and once again consider the Beltrami equation here.

Definition 6. Let Ω be a smooth domain in S^{n-1} and let $q : \Omega \rightarrow Cl_n$ be a measurable function.

Then for $f \in W^{2,1}(\Omega, Cl_n)$, the spherical Clifford Beltrami equation is given by

$$(5.1) \quad \Gamma_{\alpha} f - q \bar{\Gamma}_{\alpha} f = 0$$

In order to study this Beltrami equation, let's consider an integral equation given by

$$f = T_{\Omega} h + \phi$$

where ϕ is in the $\ker \Gamma_{\alpha}(\Omega)$ and $h = \Gamma_{\alpha} f$. Then applying $\bar{\Gamma}_{\alpha}$ on both sides of the integral equation we get

$$\bar{\Gamma}_{\alpha} f = \bar{\Gamma}_{\alpha} T_{\Omega} h + \bar{\Gamma}_{\alpha} \phi = \pi_{\alpha, S^{n-1}} h + \tilde{\phi}$$

with $\tilde{\phi} = \bar{\Gamma}_{\alpha} \phi$ and solving for h , we get

$$(5.2) \quad h = \Gamma_{\alpha} f = q \left(\pi_{\alpha, S^{n-1}} h + \tilde{\phi} \right)$$

We now consider the two equations (5.1) and (5.2). The solvability of one is the solvability of the other.

To study the solvability of (5.2), we consider the mapping :

$$h \mapsto q\pi_{\alpha, S^{n-1}}h, \quad \text{with } \|q\| < 1$$

which is a contraction map and hence it has a fixed point which is going to be a solution to (5.2). Therefore the Beltrami equation has a solution.

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