# CLIFFORD ANALYSIS ON ORLICZ-SOBOLEV SPACES

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ABSTRACT. In this article we develop few of the analogous theoretical results of Clifford analysis over Orlicz-Sobolev spaces and study mapping properties of the Dirac operator $D = \sum_{j=1}^{n} e_j \partial_{x_j}$ and the Teodorescu transform $\tau_\Omega$ over these function spaces. We also get analogous decomposition results $L^{\psi}(\Omega, Cl_n) = A^{\psi}(\Omega, Cl_n) \dotplus \overline{D}\left(W_0^{1,\psi}(\Omega, Cl_n)\right)$ of Clifford valued Orlicz spaces and the generalized Orlicz - Sobolev spaces $W^{k,\psi}(\Omega, Cl_n) = A^{k,\psi}(\Omega, Cl_n) \dotplus \overline{D}\left(W_0^{k+1,\psi}(\Omega, Cl_n)\right)$ where $\psi$ is an Orlicz function and $k \in \mathbb{N} \cup \{0\}$.

## 1. **Introduction**

*Clifford* analysis is a theoretical study of *Clifford* valued functions that are null solutions to the *Dirac* or *Dirac like* differential operators and their applications over the regular continuous function spaces $C^k(\Omega, Cl_n)$, *Lipschitz* spaces $C^{k,\lambda}(\Omega, Cl_n)$ and over *Sobolev* and *Slobodeckji* spaces $W^{k,p}(\Omega, Cl_n)$, $W^{k+\lambda,p}(\Omega, Cl_n)$ respectively for $0 < \lambda < 1$. The latter spaces are the right viable search spaces for solutions to most partial differential equations where we seek functions that are weakly differentiable as regular functions are scarce. All available literatures are done over function spaces I have indicated and the domain $\Omega$ in most cases is a bounded or unbounded but smooth region in *Euclidean* spaces $\mathbb{R}^n$ or a manifold in $\mathbb{R}^n$ or domain manifold in $\mathbb{C}^n$ with being *Lipschitz*, the minimally smoothness condition. In this paper we look at some analogous results of *Clifford* analysis over $Cl_n$-valued Orlicz and Orlicz - *Sobolev* spaces such as $L^{\psi}(\Omega, Cl_n)$ and $W^{k,\psi}(\Omega, Cl_n)$ where $\psi$ is an *Orlicz* or *Young* function.

Let $\{e_j : j = 1, 2, ..., n\}$ be an orthonormal basis for $\mathbb{R}^n$ that is equipped with an inner product so that

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \tag{1.1}$$

where $\delta_{ij}$ is the *Kronecker delta*. The *inner product* defined satisfies an anti commutative relation

$$x^2 = -\|x\|^2 \tag{1.2}$$

and with this *inner product,* $\mathbb{R}^n$ generates a $2^n$-dimensional non commutative algebra called *Clifford* algebra which is denoted by $Cl_n$.

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**Basis for** $Cl_n$ : The family

$$\{e_A : A \subset \{1 < 2 < 3 < ... < n\}\}$$

is a basis for the algebra. The object $e_0$ used above is the identity element of the *Clifford* algebra $Cl_n$.

**Representation of elements of** $Cl_n$: Every element $a \in Cl_n$ is represented by

$$a = \sum e_A a_A \tag{1.3}$$

where $a_A$ is a real number for each $A$.

**Embedding** : By identifying $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ with $\sum_{j=1}^n e_j x_j$ of $Cl_n$ we have an embedding

$$\mathbb{R}^n \hookrightarrow Cl_n$$

**Clifford conjugation:** $\overline{a}$ of a *Clifford* element $a = \sum e_A a_A \in Cl_n$ is defined as:

$$\overline{a} = \sum \overline{e}_A a_A$$

where

$$\overline{e}_A = \overline{e_{j_1} ... e_{j_r}} = (-1)^r e_{j_r} ... e_{j_1}$$

with particulars:

$$\overline{e}_j = -e_j, \quad e_j^2 = -1$$

for $i, j = 1, 2, ..., n$ and for

$$i \neq j : \overline{e_i e_j} = (-1)^2 e_j e_i = e_j e_i$$

**Definition 1.** *(Clifford norm) For* $a = \sum e_A a_A \in Cl_n$ *we define the Clifford norm of* $a$ *by*

$$\|a\|_{Cl_n} = ((a\overline{a})_0)^{\frac{1}{2}} = \left(\sum_A a_A^2\right)^{\frac{1}{2}} \tag{1.4}$$

*where* $(a)_0$ *is the real part of* $a\overline{a}$*.*

The Clifford norm $\|.\|_{Cl_n}$ satisfies the inequality:

$$\|ab\|_{Cl_n} \leq c(n) \|a\|_{Cl_n} \|b\|_{Cl_n} \tag{1.5}$$

with $c(n)$ a dimensional constant.

**Kelvin inversion:** Each non zero element $x \in \mathbb{R}^n$ has an inverse given by :

$$x^{-1} = \frac{\overline{x}}{\|x\|_{Cl_n}^2} \tag{1.6}$$

$\lessdot$ In this paper $\Omega$ is a bounded and smooth domain of $\mathbb{R}^n$ with at least a $C^1$ - hypersurface boundary.

**Function representation:** A $Cl_n$- valued function $f : \Omega \longrightarrow Cl_n$ has a representation:

$$f = \sum_{A} e_A f_A \tag{1.7}$$

where $f_A : \Omega \longrightarrow \mathbb{R}$ is a real valued component or section of $f$.

**Definition 2.** *Let $f \in C^1(\Omega) \cap C(\overline{\Omega})$, we define the Dirac derivative of $f$ by*

$$Df(x) = \sum_{j=1}^{n} e_j \partial_{x_j} f(x) \tag{1.8}$$

*A function $f : \Omega \longrightarrow Cl_n$ is called left monogenic or left Clifford analytic over $\Omega$ if*

$$Df(x) = 0, \ \forall x \in \Omega$$

*and likewise it is called right monogenic over $\Omega$ if*

$$f(x)D = \sum_{j=1}^{n} \partial_{x_j} f(x) e_j = 0, \ \forall x \in \Omega$$

An example of both left and right *monogenic* function defined over $\mathbb{R}^n \backslash \{0\}$ is given by

$$\Phi(x) = \frac{\overline{x}}{\omega_n \|x\|_{Cl_n}^n}$$

where $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

The function $\Phi$ is also a fundamental solution to the Dirac operator $D$ and we define integral transforms as convolutions of $\Phi$ with functions of some spaces below.

**Definition 3.** *Let $f \in C^1(\Omega, Cl_n) \cap C(\overline{\Omega}, Cl_n)$. We define two integral transforms as follow:*

(1.9)

*Teodorescu or Cauchy transform :* $$\zeta_\Omega f(x) = \int_\Omega \Phi(y-x) f(y) d\Omega_y = (\Phi * f)(x) \ \ , x \in \Omega$$

(1.10)

*Feuter transform :* $$\xi_{\partial\Omega} f(x) = \int_{\partial\Omega} \Phi(y-x) \upsilon(y) f(y) d\partial\Omega_y = (\Phi * \upsilon f)(x) \ \ , \ \ x \notin \partial\Omega$$

*where $\upsilon(y)$ is a unit normal vector pointing outward at $y \in \partial\Omega$ and "$*$" is a convolution.*

These transforms will also be extended to hold over Sobolev spaces $W^{k,p}(\Omega, Cl_n)$ by continuity and denseness arguments.

## 2. $\mathbf{Cl}_n$−Valued Orlicz and Orlicz – Sobolev – Slobodeckji Spaces

The function spaces we use in this paper are Cifford algebra valued Orlicz-Sobolev - Slobodeckji spaces. We therefore start with the definition of these spaces.

**Definition 4.** *A function $\psi : [0, \infty) \longrightarrow [0, \infty)$ is said to be an Orlicz function if $\psi(0) = 0$, $\lim_{x\longrightarrow\infty} \psi(x) = \infty$ and $\psi \nearrow$ and convex on its domain.*

*An example of such a function is :$\psi(x) =| x |^2$ and $\psi(x) =| x |^p$ for $1 < p < \infty$.*

**Definition 5.** *Let $\psi : [0, \infty) \longrightarrow [0, \infty)$ be an Orlicz function. A measurable, locally integrable function $f \in L_{loc}(\Omega, \mathbb{R})$ is said to belong to the Orlicz space $L^\psi(\Omega, \mathbb{R})$ if*

$$\exists\beta > 0 : \int_\Omega \psi\left(\frac{| f(x) |}{\beta}\right) d\Omega_x < \infty$$

*We thus define the Orlicz space $L^\psi(\Omega, \mathbb{R})$ as*

$$L^\psi(\Omega, \mathbb{R}) = \{f \in L_{loc}(\Omega, \mathbb{R}) : \exists\beta > 0 : \int_\Omega \psi\left(\frac{| f(x) |}{\beta}\right) d\Omega_x < \infty\}$$

*with a norm called Luxembourg norm defined as :*

$$\|f\|_{L^\psi(\Omega,\mathbb{R})} = \inf\{\beta > 0 : \int_\Omega \psi\left(\frac{| f(x) |}{\beta}\right) d\Omega_x \leq 1\} \tag{2.1}$$

The *Orlicz* power functions $\psi(x) =| x |^p$ for $1 < p < \infty$ provide the usual *Lebesgue spaces* $L^p(\Omega)$.

The theme here is to work Clifford analysis over such function spaces and develop analogous results we have on the usual regular, Lebesgue and Sobolev spaces. We start by defining how Clifford valued functions be in Orlicz spaces.

**Definition 6.** *A $Cl_n$-valued measurable and locally integrable function $f = \sum_A e_A f_A$ over $\Omega$ is said to be in the Orlicz space*

$$f \in L^\psi(\Omega, Cl_n) \Leftrightarrow f_A \in L^\psi(\Omega, \mathbb{R})$$

*with Clifford-Luxembourg norm:*

$$\|f\|_{L^\psi(\Omega,Cl_n)} = \sum_A \|f_A\|_{L^\psi(\Omega,\mathbb{R})} \tag{2.2}$$

The *Clifford-Luxembourg norm* of $f$ is defined interns of the *Luxembourg norm* of component real valued functions $f_A$.

We next define the $Cl_n-$ valued Orlicz-Sobolev spaces.

**Definition 7.** *Let $\psi$ be an Orlicz function and $k \in \mathbb{N} \cup \{0\}$. We define the Orlicz-Sobolev space $W^{k,\psi}(\Omega, Cl_n)$ as*

$$W^{k,\psi}(\Omega, Cl_n) = \{f \in L_{loc}(\Omega, Cl_n) : (\forall A)(\exists\beta_A > 0) : \sum_{0\leq|\alpha|\leq k} \int_\Omega \psi\left(\frac{| D^\alpha f_A(x) |}{\beta_A}\right) d\Omega_x < \infty\}$$

*with norm (Clifford-Luxembourg )*

$$\| f \|_{W^{k,\psi}(\Omega,Cl_n)} = \sum_A \sum_{0\leq|\alpha|\leq k} \| f_A \|_{L^{\psi}(\Omega,\mathbb{R})} = \sum_A \| f_A \|_{W^{k,\psi}(\Omega,\mathbb{R})} \tag{2.3}$$

*where*

$$\| f_A \|_{W^{k,\psi}(\Omega,\mathbb{R})} := \inf\{\beta_A > 0 : \sum_{0\leq|\alpha|\leq k} \int_\Omega \psi\left(\frac{| D^\alpha f_A(x) |}{\beta_A}\right) d\Omega_x \leq 1\}$$

*When $k = 0$ we have $L^\psi(\Omega, Cl_n)$ and*

$$f \in L^\psi(\Omega, Cl_n) \Longleftrightarrow f_A \in L^\psi(\Omega, \mathbb{R})$$

*with*

(2.4)

$$\| f \|_{L^\psi(\Omega,Cl_n)} = \sum_A \inf\{\lambda_A > 0 : \int_\Omega \psi\left(\frac{| f_A(x) |}{\lambda_A}\right) d\Omega_x \leq 1\} = \sum_A \| f_A \|_{L^\psi(\Omega,\mathbb{R})}$$

*We also define traceless Sobolev spaces as*

$$W_0^{k,\psi}(\Omega, Cl_n) := \{f \in W^{k,\psi}(\Omega, Cl_n) : f_{|\partial\Omega} = \underset{1\leq|\alpha|\leq k-1}{D^\alpha f_{|\partial\Omega}} = 0\}$$

The generalized Orlicz-Slobodeckji spaces are defined as

**Definition 8.** *The Orlicz - Slobodeckji spaces*

$$\widetilde{W}^{k-1,\psi,\psi}(\partial\Omega, Cl_n) := \{g = \tau f : f \in W^{k,\psi}(\Omega, Cl_n)\}$$

*with associated norm :*

$$\begin{aligned} \|g\|_{\widetilde{W}^{k-1,\psi,\psi}(\partial\Omega,Cl_n)} &= \sum_{\|\alpha\|\leq k-1} \int_{\partial\Omega} \psi\left(\frac{|(D^\alpha g|)|}{\lambda}\right) d\partial\Omega_x \\ &\quad + \sum_{\|\alpha\|=k-1} \int_{\partial\Omega}\int_{\partial\Omega} \psi\left(\frac{|D^\alpha g(x) - D^\alpha g(y)|}{\lambda|x-y|}\right) |x-y|^{2-n} d\partial\Omega_x d\partial\Omega_y \end{aligned}$$

*when $k = 1$, we have*

$$\widetilde{W}^{0,\psi,\psi}(\partial\Omega, Cl_n) = L^{\psi,\psi}(\partial\Omega, Cl_n)$$

*These Orlicz-Slobodeckji spaces are analogues of the Sobolev-Slobodeckji spaces*

$$W^{k-\frac{1}{p},p}(\partial\Omega, Cl_n) := \{g = \tau f : f \in W^{k,p}(\Omega, Cl_n)\}$$

*for $k \in \mathbb{N}$.*

**Proposition 1.** *The Slobodeckji space $W^{1-\frac{1}{p},p}(\partial\Omega)$ with $\lambda = 1-\frac{1}{p}$ so that $\lfloor\lambda\rfloor = 0$ and $\{\lambda\} = 1-\frac{1}{p}$ and for $f \in W^{1-\frac{1}{p},p}(\partial\Omega)$ we have*

$$\| f \|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \| f \|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \left(\frac{|f(x) - f(y)|}{|x-y|}\right)^p |x-y|^{2-n} d\partial\Omega_x d\partial\Omega_y\right)^{\frac{1}{p}}$$

*Proof.* The proof is short and straight forward by considering $\{\lambda\} = 1 - \frac{1}{p}$, $[\lambda] = 0$ so that the singularity exponent of the integrand wiil be

$$\begin{aligned} |x-y|^{-(\dim(\partial\Omega)+\{\lambda\}p)} &= |x-y|^{-(n-1+\{\lambda\}p)} \\ &= |x-y|^{-\left(n-1+\left(1-\frac{1}{p}\right)p\right)} \\ &= |x-y|^{-(n-1+p-1)} \\ &= \frac{|x-y|^{2-n}}{|x-y|^{p}} \end{aligned}$$

which provides the factor expression of the integrand of the right term of the right hand side of the two summands of the norm and the actual norm follows form the definition of norm of Slobodeckji space $W^{\lambda,p}(\partial\Omega)$.

□

**Proposition 2.** *The Orlicz-Slobodeckji space $L^{\psi,\psi}(\partial\Omega, Cl_n)$ has the following norm: for $f \in L^{\psi,\psi}(\partial\Omega, Cl_n)$,*

$$\| f \|_{L^{\psi,\psi}(\partial\Omega)} = \| f \|_{L^{\psi}(\partial\Omega, Cl_n)} + \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \psi\left(\frac{|f(x)-f(y)|}{\lambda|x-y|}\right)|x-y|^{2-n}\, d\partial\Omega_x d\partial\Omega_y$$

*with $\lambda > 0$.*

## 3. Mapping Properties of $\mathbf{D}, \boldsymbol{\zeta}_{\boldsymbol{\Omega}}$ and $\boldsymbol{\zeta}_{\partial\Omega}$

The three operators, the Dirac operator $D$, the Teodorescu or Cauchy transform $\zeta_\Omega$ and the Feuter transform $\boldsymbol{\zeta}_{\partial\Omega}$ keep integrability invariant but change regularity (smoothness) over Sobolev spaces in the following ways:

**Proposition 3.** *The Dirac operator $D : W^{k,\psi}(\Omega, Cl_n) \longrightarrow W^{k-1,\psi}(\Omega, Cl_n)$ with*

$$\|Df\|_{W^{k-1,\psi}(\Omega,Cl_n)} \leq \gamma \|f\|_{W^{k,\psi}(\Omega,Cl_n)}$$

*for $\gamma = \gamma(n, \psi, \Omega)$ a positive constant.*

*Proof.* Let $f \in W^{k,\psi}(\Omega, Cl_n)$. We need to show that

$$\begin{aligned} \|Df\|_{W^{k-1,\psi}(\Omega,Cl_n)} &= \sum_{0\leq|\alpha|\leq k-1} \| D^{\alpha}(Df) \|_{L^{\psi}(\Omega,Cl_n)} \\ &= \sum_{0\leq|\beta|\leq k} \| D^{\beta} f \|_{L^{\psi}(\Omega,Cl_n)} \\ &\leq \gamma \| f \|_{W^{k,\psi}(\Omega,Cl_n)} \end{aligned}$$

□

**Proposition 4.** $D : L^{\psi}(\Omega) \longrightarrow W^{-1,\psi}(\Omega)$ *where $\psi$ is an Orlicz function.*

*Proof.* Let $f \in L^{\psi}(\Omega, Cl_n)$. Then

$$\|Df\|_{W^{-1,\psi}(\Omega)} = \sup\{\frac{|\langle Df, g\rangle|}{\|g\|_{W_0^{1,\psi*}(\Omega)}} : g \neq 0, g \in W_0^{1,\psi*}(\Omega)\}$$

for $\psi$ and $\psi*$ are conjugate Orlicz functions.

But

$$\begin{aligned} |\langle Df, g\rangle| &= |\langle f, Dg\rangle| \leq \|f\|_{L^{\psi}(\Omega)}\|Dg\|_{L^{\psi*}(\Omega)} \\ &\leq \|f\|_{L^{\psi}(\Omega)}\|g\|_{W_0^{1,\psi*}(\Omega)} \end{aligned}$$

Thus by the Cauchy-Schwartz inequality we have

$$\frac{|\langle Df, g\rangle|}{\|g\|_{W_0^{1,\psi*}(\Omega)}} \leq \frac{\|f\|_{L^{\psi}(\Omega)}\|g\|_{W_0^{1,\psi*}(\Omega)}}{\|g\|_{W_0^{1,\psi*}(\Omega)}} = \|f\|_{L^{\psi}(\Omega)}$$

Therefore

$$\begin{aligned} \|Df\|_{W^{-1,\psi}(\Omega)} &= \sup\{\frac{|\langle Df, g\rangle|}{\|g\|_{W_0^{1,\psi*}(\Omega)}} : g \neq 0,\ g \in W_0^{1,\psi*}(\Omega)\} \\ &\leq \sup\{\frac{\|f\|_{L^{\psi}(\Omega)}\|g\|_{W_0^{1,\psi*}(\Omega)}}{\|g\|_{W_0^{1,\psi*}(\Omega)}} : g \neq 0,\ g \in W_0^{1,\psi*}(\Omega)\} \\ &= \|f\|_{L^{\psi}(\Omega)} \end{aligned}$$

□

**Proposition 5.** *Let $k \in \mathbb{N} \cup \{0\}$ and $\psi$ be an Orlicz function. Then there exists a positive constant $\beta = \beta(n, \psi, \Omega)$ such that*

$$\zeta_{\Omega} : W^{k,\psi}(\Omega, Cl_n) \longrightarrow W^{k+1,\psi}(\Omega, Cl_n) \tag{3.1}$$

*with*

$$\|\zeta_{\Omega} f\|_{W^{k+1,\psi}(\Omega,Cl_n)} \leq \beta\|f\|_{W^{k,\psi}(\Omega,Cl_n)}$$

*Proof.* Let $f \in W^{k,\psi}(\Omega, Cl_n)$. Then clearly $\zeta_{\Omega} f \in W^{k+1,\psi}(\Omega, Cl_n)$ as $D\zeta_{\Omega} f = f$ from Borel-Pompeiu relation and we have norm estimates

$$\begin{aligned} \|\zeta_{\Omega} f\|_{W^{k+1,\psi}(\Omega,Cl_n)} &= \sum_{0\leq|\alpha|\leq k+1} \| D^{\alpha}\zeta_{\Omega} f \|_{L^{\psi}(\Omega,Cl_n)} \\ &= \sum_{0\leq|\beta|\leq k} \| D^{\beta}(D\zeta_{\Omega} f) \|_{L^{\psi}(\Omega,Cl_n)} \\ &= \sum_{0\leq|\beta|\leq k} \| D^{\beta} f \|_{L^{\psi}(\Omega,Cl_n)} \\ &\leq \gamma \| f \|_{W^{k,\psi}(\Omega,Cl_n)} \end{aligned}$$

□

**Proposition 6.** *We also have the mapping properties of the boundary Feuter integral $\xi_{\partial\Omega}$ and the trace operator $\tau$:*
($i$) *The Feuter transform :*

$$\xi_{\partial\Omega} : \widetilde{W}^{k-1,\psi,\psi}\left(\partial\Omega, Cl_n\right) \longrightarrow W^{k,\psi}\left(\Omega, Cl_n\right) \tag{3.2}$$

*with*

$$\|\xi_{\partial\Omega} f\|_{W^{k,\psi}(\Omega,Cl_n)} \leq \theta \|f\|_{\widetilde{W}^{k-1,\psi,\psi}(\partial\Omega,Cl_n)}$$

*and*
($ii$) *the trace operator :*

$$\tau : W^{k,\psi}(\Omega, Cl_n) \longrightarrow \widetilde{W}^{k-1,\psi,\psi}\left(\partial\Omega, Cl_n\right) \tag{3.3}$$

*with*

$$\begin{aligned}
\| \tau f \|_{\widetilde{W}^{k-1,\psi,\psi}(\partial\Omega,Cl_n)} &= \| \tau f \|_{W^{k-1,\psi}(\partial\Omega,Cl_n)} \\
&\quad + \sum_{|\alpha|=k-1} \int_{\partial\Omega}\int_{\partial\Omega} \psi\left(\frac{|D^\alpha \tau f(x) - D^\alpha \tau f(y)|}{\lambda\,|x-y|}\,|x-y|^{2-n}\, d\partial\Omega_x d\partial\Omega_y\right) \\
&\leq \theta_1 \| f \|_{W^{k,\psi}(\Omega,Cl_n)} \\
&\quad +\theta_2 \sum_{|\alpha|=k-1} \int_{\Omega}\int_{\Omega} \psi\left(\frac{|D^\alpha f(x) - D^\alpha f(y)|}{\lambda\,|x-y|}\,|x-y|^{2-n}\, d\partial\Omega_x d\partial\Omega_y\right) \\
&= \theta \left( \begin{array}{c} \| f \|_{W^{k,\psi}(\Omega,Cl_n)} \\ + \sum_{|\alpha|=k-1} \int_{\Omega}\int_{\Omega} \psi\left(\frac{|D^\alpha f(x) - D^\alpha f(y)|}{\lambda|x-y|}\,|x-y|^{2-n}\, d\partial\Omega_x d\partial\Omega_y\right) \end{array} \right) \\
&= \theta \| f \|_{W^{k,\psi}(\Omega,Cl_n)}
\end{aligned}$$

*where $\theta_1, \theta_2$ are quantities of $(n, \psi, \Omega)$ and $\delta = \delta\,(n, \psi, \Omega)$ with $\theta = \max\{\theta_1, \theta_2\}$*

**Proposition 7.** *The composition $\xi_{\partial\Omega} \circ \tau$ preserves regularity of a function in a Sobolev space.*

*Proof.* Indeed the trace operator $\tau$ makes a function to loose a regularity exponent of *one* when acted along the boundary of the domain keeping integrability index unchanged. But the boundary or *Feuter* integral $\xi_{\partial\Omega}$ augments the regularity exponent of a function defined on the boundary by an exponent that is lost by the trace operator and therefore the composition operator $\xi_{\partial\Omega} \circ \tau$ preserves or restores the regularity exponent of a function in a Sobolev space. □

The following proposition is what I call it the trinity of Clifford analysis based on the relationship that connects $I, \xi_{\partial\Omega}$ and $\zeta_\Omega$ where $I$ is the identity operator.

**Proposition 8.** *(Borel-Pompeiu ) Let* $f \in W^{k,\psi}(\Omega, Cl_n)$. *Then*

$$f = \xi_{\partial\Omega}\tau f + \zeta_\Omega Df$$

*Proof.* The proof can be done either through Gauss theorem or integration by parts shown below first for a function $f \in C^\infty(\Omega, Cl_n) \cap W^{k,\psi}(\Omega, Cl_n)$

$$\int_\Omega \Phi(x-y)\, Df(y) d\Omega_y = \int_{\partial\Omega} \Phi(x-y)\, n(y) f(y) d\partial\Omega_y - \int_\Omega D\Phi(x-y)\, f(y) d\Omega_y$$

But

$$\int_\Omega D\Phi(x-y)\, f(y) d\Omega_y = \int_\Omega \delta(x-y)\, f(y) d\Omega_y = f(x)$$

where $\delta$ here is the Dirac-delta (impulse) distribution and rearranging terms we get the result.

Then since $C^\infty(\Omega, Cl_n) \cap W^{k,\psi}(\Omega, Cl_n)$ is dense in $W^{k,\psi}(\Omega, Cl_n)$ and by continuity arguments for $f \in W^{k,\psi}(\Omega, Cl_n)$ we get a sequence $\{f_n : n \in \mathbb{N}\} \subseteq C^\infty(\Omega, Cl_n) \cap W^{k,\psi}(\Omega, Cl_n)$ such that $f_n \longrightarrow f$ in $W^{k,\psi}(\Omega, Cl_n)$ sense and that completes the proof. □

**Corollary 1.** $(i)$ *If* $f \in W_0^{k,\psi}(\Omega, Cl_n)$, *then*

$$\begin{aligned} f(x) &= \int_\Omega \Phi(x-y)\, Df(y) d\Omega_y \\ &= \xi_\Omega Df \end{aligned}$$

*That is* $D$ *is a right inverse for* $\zeta_\Omega$ *and* $\zeta_\Omega$ *is a left inverse for* $D$ *over traceless spaces.*

$(ii)$ *If* $f$ *is monogenic function over* $\Omega$, *then*

$$\begin{aligned} f(x) &= \int_{\partial\Omega} \Phi(x-y)\, n(y) f(y) d\partial\Omega_y \\ &= \xi_{\partial\Omega}\tau f \end{aligned}$$

*Therefore monogenic functions are always Cauchy transforms of their traces over the boundary.*

*Proof.* The proof follows from the above Borel-Pompeiu result. But a further note from $(i)$ and $(ii)$ of the corollary is that a traceless monogenic function is a null function. □

## 4. **Decomposition Results**

In this section we present two decomposition results, one for the $Cl_n-$ valued Orlicz space $L^\psi(\Omega, Cl_n)$ and for the generalized Orlicz-Sobolev space $W^{k,\psi}(\Omega, Cl_n)$.

But first,

**Definition 9.** *Let* $\psi$ *be an Orlicz function, we define*

$(i)$ *The* $\psi-$ *Orlicz - Bergman space*

$$A^\psi(\Omega, Cl_n) := \{f \in L^\psi(\Omega \longrightarrow Cl_n) : Df = 0 \text{ on } \Omega\} = L^\psi(\Omega, Cl_n) \cap \ker D$$

*and*

(*ii*) *The generalized $\psi-$ Orlicz-Sobolev - Bergamn space*

$$A^{k,\psi}(\Omega, Cl_n) := W^{k,\psi}(\Omega, Cl_n) \cap \ker D$$

The first decomposition result for the Orlicz-Sobolev space:

**Proposition 9.** *Let $\psi : [0,\infty) \longrightarrow [0,\infty)$ be an Orlicz function. Then we have the direct decomposition of the Orlicz space*

$$L^{\psi}(\Omega, Cl_n) = A^{\psi}(\Omega, Cl_n) \oplus \overline{D}\left(W_0^{1,\psi}(\Omega, Cl_n)\right)$$

*where $A^{\psi}(\Omega, Cl_n)$ is the $\psi-$Orlicz - Bergman space over $\Omega$.*

*Proof.* Let

$$f \in A^{\psi}(\Omega, Cl_n) \oplus \overline{D}\left(W_0^{1,\psi}(\Omega, Cl_n)\right)$$

Then $Df = 0$ and $f = \overline{D}g$ for some $g \in W_0^{1,\psi}(\Omega, Cl_n)$. But then

$$Df = D(\overline{D}g) = \Delta g_{|_{W_0^{1,\psi}(\Omega, Cl_n)}} = \Delta_0 g = 0$$

and from invertibility of $\Delta_0 : W_0^{1,\psi}(\Omega, Cl_n) \longrightarrow Cl_n$, we see that $g = 0$. Therefore $f \equiv 0$ which implies

$$A^{\psi}(\Omega, Cl_n) \oplus \overline{D}\left(W_0^{1,\psi}(\Omega, Cl_n)\right) = \{0\}$$

Again to show that every element $f \in L^{\psi}(\Omega, Cl_n)$ is a sum of elements form the summand spaces $A^{\psi}(\Omega, Cl_n)$ and $\overline{D}\left(W_0^{1,\psi}(\Omega, Cl_n)\right)$.

Let $f \in L^{\psi}(\Omega, Cl_n)$ and take $\eta = \Delta_0^{-1}Df \in W_0^{1,\psi}(\Omega, Cl_n)$, define a function $g := f - \eta$. Then $Dg = D(f - \eta) = 0$ which implies

$$g \in KerD \cap L^{\psi}(\Omega, Cl_n) = A^{\psi}(\Omega, Cl_n)$$

Thus

$$f = g \dotplus \eta \in A^{\psi}(\Omega, Cl_n) \oplus \overline{D}\left(W_0^{1,\psi}(\Omega, Cl_n)\right)$$

where $\dotplus$ is used for elemental direct sum and that proves the proposition. □

The second decomposition result for the generalized Orlicz-Sobolev space:

**Proposition 10.** *The Clifford valued Sobolev space $W^{k,\psi}(\Omega, Cl_n)$ has a similar direct decomposition*

$$W^{k,\psi}(\Omega, Cl_n) = A^{k,\psi}(\Omega, Cl_n) \dotplus \overline{D}\left(W_0^{k+1,\psi}(\Omega, Cl_n)\right)$$

*where $A^{k,\psi}(\Omega, Cl_n)$ is the generalized $\psi$- Orlicz-Sobolev - Bergman space over $\Omega$.*

*Proof.* The proof follows the same argument as above. □

## 5. First Order EllipticBVP

Here we look at first order elliptic boundary value problems of the Dirac operator and provide norm estimates of a solution in terms of norms of the input data.

**Proposition 11.** *Let* $f \in W^{k-1,\psi}(\Omega, Cl_n)$ *and* $g \in \widetilde{W}^{k-1,\psi,\psi}(\partial\Omega, Cl_n)$ *for* $k \geq 1$. *Then the first order elliptic BVP:*

$$\left\{ \begin{array}{l} Du = f \quad in\ \Omega \\ \tau u = g \ \ on\ \partial\Omega \end{array} \right. \tag{5.1}$$

*has a solution* $u \in W^{k,\psi}(\Omega, Cl_n)$ *given by*

$$u(x) = \xi_{\partial\Omega} g + \zeta_\Omega f$$

*Proof.* The proof follows from the Borel-Pompeiu relation. As to where exactly $u$ and $g$ belong, we make the argument : $f$ is in $W^{k-1,\psi}(\Omega, Cl_n)$ and hence from the mapping property of $D$, we have $u$ to be a function in $W^{k,\psi}(\Omega, Cl_n)$.

Also from the mapping property of the trace operator $\tau$ we have

$$\tau u = u|_{\partial\Omega} = g \in \widetilde{W}^{k-1,\psi,\psi}(\partial\Omega, Cl_n)$$

□

**Proposition 12.** *The solution* $u \in W^{k,\psi}(\Omega, Cl_n)$ *of the elliptic BVP* (5.1) *has a norm estimate :*

$$\begin{aligned} \|u\|_{W^{k,\psi}(\Omega, Cl_n)} \ \leq \ \gamma_1 & \left( \begin{array}{c} \sum\limits_{\|\alpha\| \leq k-1} \int_{\partial\Omega} \psi\left(\frac{|(D^\alpha g|)|}{\lambda}\right) d\partial\Omega_x \\ + \sum\limits_{\|\alpha\|=k-1} \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \psi\left(\frac{|D^\alpha g(x) - D^\alpha g(y)|}{\lambda|x-y|}\right) |x-y|^{2-n} \, d\partial\Omega_x d\partial\Omega_y \end{array} \right) \\ & + \gamma_2 \left( \sum_{\|\alpha\|=k-1} \int_\Omega \psi\left(\frac{|D^\alpha f(x)|}{\lambda}\right) d\Omega_x \right) \end{aligned}$$

*where* $\gamma_1, \gamma_2$ *are constants the depend on p,n and* $\Omega$.

*Proof.* Clearly from the mapping properties of $D$, $\zeta_\Omega$, $\tau$ and $\xi_{\partial\Omega}$ $D$ and because $g \in \widetilde{W}^{k-1,\psi,\psi}(\partial\Omega, Cl_n)$ and $f \in W^{k-1,,\psi}(\Omega, Cl_n)$ we have

$$u \in W^{k,\psi}(\Omega, Cl_n)$$

From the Borel-Pompeiu theorem we have the solution $u$ given by:

$$u(x) = \xi_{\partial\Omega} g + \zeta_\Omega f$$

Now the of the solution $u$ can be estimated in the following sequence of inequalities:

$$\begin{aligned}
\|u\|_{W^{k,p}(\Omega,Cl_n)} &= \|\xi_{\partial\Omega} g + \zeta_\Omega f\|_{W^{k,p}(\Omega,Cl_n)} \\
&\leq \|\xi_{\partial\Omega} g\|_{W^{k,p}(\Omega,Cl_n)} + \|\zeta_\Omega f\|_{W^{k,p}(\Omega,Cl_n)} \\
&\leq \gamma_1 \|g\|_{\widetilde{W}^{k-1,\psi,\psi}(\partial\Omega,Cl_n)} + \gamma_2 \|f\|_{W^{k-1,p}(\Omega,Cl_n)} \\
&= \gamma_1 \left( \begin{array}{c} \sum_{\|\alpha\|\leq k-1} \int_{\partial\Omega} \psi\left(\frac{|D^\alpha g(x)|}{\lambda}\right) d\partial\Omega_x \\ + \sum_{\|\alpha\|=k-1} \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \psi\left(\frac{|D^\alpha g(x) - D^\alpha g(y)|}{\lambda|x-y|}\right) |x-y|^{2-n}\, d\partial\Omega_x d\partial\Omega_y \end{array} \right) \\
&\quad + \gamma_2 \left( \sum_{\|\alpha\|=k-1} \int_\Omega \psi\left(\frac{|D^\alpha f(x)|}{\lambda}\right) d\Omega_x \right) \\
&= \gamma_1 \left( \begin{array}{c} \sum_{\|\alpha\|\leq k-1} \int_{\partial\Omega} \psi\left(\frac{|D^\alpha g(x)|}{\lambda}\right) d\partial\Omega_x \\ + \sum_{\|\alpha\|=k-1} \int\limits_{\partial\Omega}\int\limits_{\partial\Omega} \psi\left(\frac{|D^\alpha g(x) - D^\alpha g(y)|}{\lambda|x-y|}\right) |x-y|^{2-n}\, d\partial\Omega_x d\partial\Omega_y \end{array} \right) \\
&\quad + \gamma_2 \left( \sum_{\|\alpha\|=k-1} \int_\Omega \psi\left(\frac{|D^\alpha f(x)|}{\lambda}\right) d\Omega_x \right)
\end{aligned}$$

The constants $\gamma_1$ and $\gamma_2$ are from the mapping properties of $\xi_{\partial\Omega}, \zeta_\Omega$ and $\tau$.

□

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