Adv. Oper. Theory 2 (2017), no. 4, 419–427
http://doi.org/10.22034/aot.1703-1135
ISSN: 2538-225X (electronic)
http://aot-math.org

ON ORTHOGONAL DECOMPOSITION OF A SOBOLEV SPACE

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Communicated by P. Aiena

Abstract. The theme of this short article is to investigate an orthogonal decomposition of the Sobolev space $W^{1,2}(\Omega)$ as

$$W^{1,2}(\Omega) = A^{2,2}(\Omega) \oplus D^2\left(W_0^{3,2}(\Omega)\right)$$

and look at some of the properties of the inner product therein and the distance defined by the inner product. We also determine the dimension of the orthogonal difference space $W^{1,2}(\Omega) \ominus \left(W_0^{1,2}(\Omega)\right)^{\perp}$ and show the expansion of Sobolev spaces as their regularity increases.

1. Introduction

This is an extension work of [1] and [2] in which the space under consideration is a Sobolev space of regularity exponent one. The change in regularity, from the Lebesgue space of regularity zero to Sobolev spaces of higher regularities, causes increase in length or norm, expansion of the space in terms of distance or separation between distinct elements and change in orthogonality.

In addition to the regular properties we develop in the decomposition process, we obtain some geometric properties of distance and apertures between non zero elements.

First, let us adopt the following notations that are used in this short article.

$\Omega := [0, 1]$,

$\oplus :=$ Direct sum for sets,

$\ominus :=$ Direct difference of sets,

Date: Received: Mar. 11, 2017; Accepted: Jun 8, 2017.

2010 *Mathematics Subject Classification.* Primary 46E35; Secondary 46C15.

Key words and phrases. Sobolev space, orthogonal decomposition, inner product, distance.

$\uplus$:= Direct sum of elements from orthogonal sets,
$D^\alpha := \frac{d^\alpha}{dx^\alpha}$ for $\alpha = 0, 1, 2,$
$C^\infty(\Omega) = \bigcup_{n=0}^\infty C^n(\Omega),$
$C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) : \text{supp} f \subseteq K \subseteq \Omega\}.$

Definition 1.1. We say that a function $h : \Omega \longrightarrow \mathbb{R}$ is a weak derivative of g of order α if $\int_\Omega h(x)\phi(x)dx = (-1)^\alpha \int_\Omega g(x)D^\alpha\phi(x)\, dx$ for all $\phi \in C_0^\infty(\Omega)$.

Clearly functions which are differentiable in the regular sense of a certain order are weakly differentiable of that order but the converse is not true in general.

Example 1.2. Let the function be given by $f(x) = \begin{cases} x - \frac{1}{2} & \frac{1}{2} \leq x \leq 1 \\ 0 & 0 \leq x \leq \frac{1}{2} \end{cases}$. Then $f \in C^0(\Omega) \setminus C^1(\Omega)$, i.e. f is continuous but not differentiable in the regular sense but weakly differentiable with weak derivative $Df = g = \begin{cases} 1 & \frac{1}{2} < x \leq 1 \\ 0 & 0 \leq x < \frac{1}{2} \end{cases}$.
Indeed,

$$\int_\Omega f\phi' dx = \int_\Omega \left(x - \frac{1}{2}\right)\phi' dx = -\int \left(x - \frac{1}{2}\right)'\phi dx = -\int_{\frac{1}{2}}^1 \phi dx = -\int_0^1 Df\phi dx.$$

Definition 1.3. The Sobolev space $W^{1,2}(\Omega)$ is defined as $\{f \in \mathcal{L}^2(\Omega) : Df \in \mathcal{L}^2(\Omega)\}$ and $W_0^{1,2}(\Omega) = \left\{f \in W^{1,2}(\Omega) : f_{|\partial\Omega} = 0\right\}$, where Df is in the sense of weak distributional derivative.

Remark 1.4. $W^{1,2}(\Omega) \subseteq \mathcal{L}^2(\Omega)$ but the converse is not true. For example, the function $f(x) = \sqrt{x} \in \mathcal{L}^2(\Omega)$ but $f \notin W^{1,2}(\Omega)$, since $f'(x)$ has a singularity at $x = 0$, where the improper integral $\int_\Omega | f'(x) |^2 dx$ diverges. That is $f' \notin \mathcal{L}^2(\Omega)$

Question: Is there a non trivial function $f \in W^{1,2}(\Omega) \ominus W_0^{1,2}(\Omega)$ and how big is $W^{1,2}(\Omega) \ominus W_0^{1,2}(\Omega)$?

Embedding. Clearly the Sobolev space $W^{1,2}(\Omega)$ is not a collection of wildly behaved generalized functions but some how well behaved functions that are more than continuous.

In fact, the space is embedded in the Hölder space C^γ for $0 \leq \gamma \leq \frac{1}{2}$, in particularly $W^{1,2}(\Omega) \hookrightarrow C^{\frac{1}{2}}(\Omega)$. Recall that the Hölder space $C^{\frac{1}{2}}(\Omega)$ is the space of functions f with property $\| f(x) - f(y) \| \leq \lambda_f \| x - y \|^{\frac{1}{2}}$ for all $x, y \in \Omega$ for some non negative constant λ_f that depends on f.

2. Inner Product and Orthogonality

The Sobolev space $W^{1,2}(\Omega)$ is an inner product space under the inner product defined as

$$\langle f, g\rangle_{W^{1,2}(\Omega)} := \int_\Omega fg + f'g' dx \tag{2.1}$$

and from this inner product a norm is defined as

$$\| f \|_{W^{1,2}(\Omega)} = \left(\langle f, f\rangle_{W^{1,2}(\Omega)}\right)^{\frac{1}{2}}. \tag{2.2}$$

Clearly one can verify the following

(1) $\| \bullet \|_{W^{1,2}(\Omega)} \geq \| \bullet \|_{\mathcal{L}^2(\Omega)}$,
(2) $\langle f, g \rangle_{W^{1,2}(\Omega)} = \langle f, g \rangle_{\mathcal{L}^2(\Omega)} + \langle f', g' \rangle_{\mathcal{L}^2(\Omega)}$.

Definition 2.1. Two functions f and g of $W^{1,2}(\Omega)$ are said to be orthogonal with respect to the inner product (2.1) defined above if and only if $\langle f, g \rangle_{W^{1,2}(\Omega)} = 0$.

Example 2.2. (1) $\langle \sin x, \sin x \rangle_{W^{1,2}(\Omega)} = \left(\int_\Omega \sin^2 x + \cos^2 x dx \right)^{\frac{1}{2}} = 1$. Hence $\| \sin x \|_{W^{1,2}(\Omega)} = 1$.
(2) $\langle \sin x, \cos x \rangle_{W^{1,2}(\Omega)} = 0$ and hence $\sin x$ and $\cos x$ are orthogonal in $W^{1,2}(\Omega)$ but not in $\mathcal{L}^2(\Omega)$.
(3) $\left\langle e^{\alpha x}, e^{\beta x} \right\rangle_{W^{1,2}(\Omega)} = 0$ for $\alpha\beta = -1$ but not in $\langle \cdot, \cdot \rangle_{\mathcal{L}^2(\Omega)}$. Hence for $\alpha\beta = -1$, $e^{\alpha x}$ and $e^{\beta x}$ are orthogonal in $W^{1,2}(\Omega)$ not in $\mathcal{L}^2(\Omega)$. In particular, for $\alpha = 1, \beta = -1$, $\langle e^x, e^{-x} \rangle_{W^{1,2}(\Omega)} = 0$.

Proposition 2.3. *For $\lambda > 0$, $f \in W^{1,2}(\Omega)$, it holds that*

$$\langle f, \lambda f \rangle_{W^{1,2}(\Omega)} = \lambda \| f \|^2_{W^{1,2}(\Omega)} .$$

Proof. $\langle f, \lambda f \rangle_{W^{1,2}(\Omega)} = \int_\Omega \lambda f^2 + \lambda (f')^2 dx = \lambda \left(\int_\Omega f^2 + (f')^2 \right) dx = \lambda \| f \|^2_{W^{1,2}(\Omega)}$. □

Proposition 2.4. *Norm is longer in $W^{1,2}(\Omega)$ than in $\mathcal{L}^2(\Omega)$, i.e.*
(1) *For $f \in W^{1,2}(\Omega)$ it holds that*

$$\| f \|_{W^{1,2}(\Omega)} \geq \| f \|_{\mathcal{L}^2(\Omega)} .$$

(2) *For $f \in W^{1,2}(\Omega) \cap C^1(\Omega)$ with $f' = \alpha f$ for some $\alpha \neq 0$ it holds that*

$$\| f \|_{W^{1,2}(\Omega)} = \sqrt{1 + \alpha^2} \| f \|_{\mathcal{L}^2(\Omega)} .$$

Proof. Let $f \in W^{1,2}(\Omega)$. Then
(1) $\| f \|_{W^{1,2}(\Omega)} = \left(\int_\Omega f^2 + f'^2 dx \right)^{\frac{1}{2}} \geq \left(\int_\Omega f^2 dx \right)^{\frac{1}{2}} = \| f \|_{\mathcal{L}^2(\Omega)}$.
(2) For $f \in W^{1,2}(\Omega) \cap C^1(\Omega)$ with $f' = \alpha f$ for some $\alpha \neq 0$, we have

$$\begin{aligned} \| f \|_{W^{1,2}(\Omega)} &= \left(\int_\Omega f^2 + f'^2 dx \right)^{\frac{1}{2}} = \left(\int_\Omega f^2 + (\alpha f)^2 dx \right)^{\frac{1}{2}} = \left(\int_\Omega f^2 + \alpha^2 f^2 dx \right)^{\frac{1}{2}} \\ &= \left(\int_\Omega (1 + \alpha^2) f^2 dx \right)^{\frac{1}{2}} = \sqrt{1 + \alpha^2} \| f \|_{\mathcal{L}^2(\Omega)} . \end{aligned}$$

□

Note that (1) follows from (2) since $\sqrt{1 + \alpha^2} \geq 1$.

An important question one can pause: which elements $f \in W^{1,2}(\Omega)$ maintain their norms of $\mathcal{L}^2(\Omega)$? The answer lies in the next proposition.

Proposition 2.5. *A function $f \in W^{1,2}(\Omega)$ which is a.e. a constant over Ω maintains its norm of $\mathcal{L}^2(\Omega)$.*

Proof. Let $f \in W^{1,2}(\Omega)$ which is a.e. a constant. That is $f' = 0$ a.e. over Ω. Then
$\| f \|_{W^{1,2}(\Omega)} = \sqrt{\langle f, f\rangle_{W^{1,2}(\Omega)}} = \sqrt{\int_\Omega (f^2 + f'^2)\, dx}$ and
$\sqrt{\int_\Omega (f^2 + f'^2)\, dx} = \| f \|_{\mathcal{L}^2(\Omega)}$ only when $\int_\Omega f'^2 dx = 0$. That is, $f' = 0$ a.e. over Ω means f is a constant a.e. □

(1) $\| e^{\pm x} \|_{W^{1,2}(\Omega)} = \sqrt{2} \| e^{\pm x} \|_{\mathcal{L}^2(\Omega)}$.
(2) $\| x \|_{W^{1,2}(\Omega)} = 2 \| x \|_{\mathcal{L}^2(\Omega)}$.

In the following proposition, we show that if a function $f \in C^1(\Omega)$ and its derivative f' have a vanishing property over the boundary of Ω, then always f and its derivative f' are orthogonal over Ω.

Proposition 2.6. *For $f \in C^1(\Omega)$ with boundary conditions $f_{|\partial\Omega} = 0$, $f'_{|\partial\Omega} = 0$, the functions f and f' are orthogonal in $W^{1,2}(\Omega)$, i.e. $\langle f, f'\rangle_{W^{1,2}(\Omega)} = 0$.*

Proof. Let $f \in C^1(\Omega)$. Then $\langle f, f'\rangle_{W^{1,2}(\Omega)} = \int_\Omega ff' + f'f''dx$ but $ff' + f'f'' = \frac{1}{2}\left((f^2)' + \left((f')^2\right)'\right) = \frac{1}{2}\left(f^2 + (f')^2\right)'$.
Thus $\int_\Omega ff' + f'f''dx = \frac{1}{2}\int_\Omega \left(f^2 + (f')^2\right)' dx = \frac{1}{2}\left(f^2 + (f')^2\right) |_0^1 = 0$. □

Remark 2.7. The converse of Proposition 2.5 does not hold true, since $\sin x$ and $\cos x$ are $W^{1,2}(\Omega)$-orthogonal but $\sin x_{|\partial\Omega} \neq 0$ and $\cos x_{|\partial\Omega} \neq 0$.

Example 2.8. For $\alpha > 1, \beta > 1$ let $f(x) = x^\alpha (x-1)^\beta$. Then from the above proposition we have $\langle f, f'\rangle_{W^{1,2}(\Omega)} = 0$.

Proposition 2.9. *Let $f, g \in W^{1,2}(\Omega)$.*

(1) $\langle f, g\rangle_{W^{1,2}(\Omega)} = 0 \implies \langle f, g\rangle_{\mathcal{L}^2(\Omega)} = -\langle f', g'\rangle_{\mathcal{L}^2(\Omega)}$.
(2) $\langle f, g\rangle_{\mathcal{L}^2(\Omega)} = 0 \implies \langle f, g\rangle_{W^{1,2}(\Omega)} = \langle f', g'\rangle_{\mathcal{L}^2(\Omega)}$.
(3) *When pair wise f and g and $f'$and g' are $\mathcal{L}^2(\Omega)$-orthogonal, then f and g are $W^{1,2}(\Omega)$-orthogonal.*

Proposition 2.10. *Let f be a function in $W^{1,2}(\Omega)$ with non zero norm and α be a non zero constant. Then* $\cos\theta\langle f, \alpha f) = \begin{cases} 1, \text{ for } \alpha > 0 \\ -1, \text{ for } \alpha < 0 \end{cases}$.

Proof. $\cos\theta\langle (f, \alpha f)_{W^{1,2}(\Omega)} = \frac{\langle f, \alpha f\rangle_{W^{1,2}(\Omega)}}{|\alpha| \|f\|^2_{W^{1,2}(\Omega)}} = \frac{\alpha \|f\|^2_{W^{1,2}(\Omega)}}{|\alpha| \|f\|^2_{W^{1,2}(\Omega)}} = \frac{\alpha}{|\alpha|} = \begin{cases} 1, \text{ for } \alpha > 0 \\ -1, \text{ for } \alpha < 0 \end{cases}$.
Thus $\alpha > 0 \implies \theta = 0$ and $\alpha < 0 \implies \theta = \pi$. □

Definition 2.11. The distance $\rho_{W^{1,2}(\Omega)}$ between two elements f and g of $W^{1,2}(\Omega)$ is given by $\rho_{W^{1,2}(\Omega)}(f, g) = \| f - g \|_{W^{1,2}(\Omega)}$.

Proposition 2.12. *For $\lambda (\neq 0) \in \mathbb{R}$,*

$$\rho_{W^{1,2}(\Omega)}(f, \lambda f) = | 1 - \lambda | \| f \|_{W^{1,2}(\Omega)} .$$

Proof.

$$\begin{aligned}\rho_{W^{1,2}(\Omega)}(f,\lambda f) &= \left(\int_\Omega (f-\lambda f)^2+(f'-\lambda f')^2\,dx\right)^{\frac{1}{2}}\\ &= \left(\int_\Omega f^2(1-\lambda)^2+f'^2(1-\lambda)^2\,dx\right)^{\frac{1}{2}}\\ &= \left((1-\lambda)^2\right)^{\frac{1}{2}}\left(\int_\Omega f^2+f'^2dx\right)^{\frac{1}{2}}\\ &= |\,1-\lambda\,|\|\,f\,\|_{W^{1,2}(\Omega)}\,.\end{aligned}$$

□

Corollary 2.13. *For $\lambda\,(\neq 0)\in\mathbb{R}$, only when $\lambda=2$ that $\rho_{W^{1,2}(\Omega)}(f,\lambda f)=\rho_{W^{1,2}(\Omega)}(f,2f)=\|\,f\,\|_{W^{1,2}(\Omega)}$.*

Corollary 2.14. *For $\lambda<0$ and $\lambda>2$, $\rho_{W^{1,2}(\Omega)}(f,\lambda f)>\|\,f\,\|_{W^{1,2}(\Omega)}$ and for $0<\lambda<2$ it holds that $\rho_{W^{1,2}(\Omega)}(f,\lambda f)<\|\,f\,\|_{W^{1,2}(\Omega)}$.*

We also have

(1) $\rho_{W^{1,2}(\Omega)}(\cos x,\sin x)=\sqrt{2}$.

(2) $\rho_{W^{1,2}(\Omega)}(e^x,e^{-x})=\frac{\sqrt{e^4-1}}{e}$.

From the assertion 2 above and from the following two propositions, we see a Sobolev space is expanding as its regularity increases.

In the next proposition we see that non zero elements are going further apart in $W^{1,2}(\Omega)$ than they were in the Hilbert space $\mathcal{L}^2(\Omega)$

Proposition 2.15. *For $f,g\in W^{1,2}(\Omega)$, it holds that $\rho_{\mathcal{L}^2(\Omega)}(f,g)\le\rho_{W^{1,2}(\Omega)}(f,g)$.*

Thus

(1) $\rho_{W^{1,2}(\Omega)}(\cos x,\sin x)=\sqrt{2}\ge\sqrt{1-\sin^2 1}=\rho_{\mathcal{L}^2(\Omega)}(\cos x,\sin x)$.

(2) $\rho_{W^{1,2}(\Omega)}(e^x,e^{-x})=\frac{\sqrt{e^4-1}}{e}\ge\frac{\sqrt{e^4-2e^2-1}}{e\sqrt{2}}=\rho_{\mathcal{L}^2(\Omega)}(e^x,e^{-x})$.

Proposition 2.16 (Generalizing Proposition 8)**.** *The Sobolev space is expanding with regularity, i.e.,*

$$\rho_{W^{(k-1),2}(\Omega)}(f,g)\le\rho_{W^{k,2}(\Omega)}(f,g)$$

for $k\in\mathbb{Z}^+,\ f,g\in W^{k,2}(\Omega)$.

Proof.

$$\begin{aligned}\rho_{W^{k,2}(\Omega)}(f,g) &= \left(\sum_{j=0}^{k}\left(f^{(j)}-g^{(j)}\right)^2\right)^{\frac{1}{2}}\ge\left(\sum_{j=0}^{k-1}\left(f^{(j)}-g^{(j)}\right)^2\right)^{\frac{1}{2}}\\ &= \rho_{W^{(k-1),2}(\Omega)}(f,g)\,,\end{aligned}$$

where $f^{(j)}=\frac{d^j}{dx^j}(f)$. □

3. Orthogonal Decomposition

We start this section with the following result.

Proposition 3.1 (Orthogonal Decomposition)**.**

$$W^{1,2}(\Omega) = A^{2,2}(\Omega) \oplus D^2\left(W_0^{3,2}(\Omega)\right), \tag{3.1}$$

where $A^{2,2}(\Omega) = \ker D^2(\Omega) \cap W^{1,2}(\Omega)$ *is the Bergman space in one dimension.*

Proof. Let $f \in W^{1,2}(\Omega)$ and let $\eta = D_0^{-4}(D^2 f)$. Define $g = f - D^2\eta$. Then clearly $g \in \ker D^2(\Omega)$ and $\eta \in W_0^{3,2}(\Omega)$. Therefore $f = g \uplus D^2\eta$. □

From the orthogonal decomposition 3.1, there are canonical orthogonal projections P and Q with $P : W^{1,2}(\Omega) \longrightarrow A^{2,2}(\Omega)$ and $Q : W^{1,2}(\Omega) \longrightarrow D^2\left(W_0^{3,2}(\Omega)\right)$ such that $P + Q = I$, where I is the identity operator.

Corollary 3.2. $PQ = QP = 0$ *and* $P^2 = P$ *and* $Q^2 = Q$

Proof. Clearly $PQ = 0 = QP$. Then the other two follow from this and the fact that $P + Q = I$ □

Example 3.3. We present few but fundamental decompositions of elementary functions

(1) For $f(x) = x$, $P(f) = f$ and $Q(f) = 0$ so that $f = f \uplus 0$.
(2) For $f(x) = x^2$, $P(f) = x - \frac{1}{6}$ and $Q(f) = x^2 - x + \frac{1}{6}$. So that

$$x^2 = \left(x - \frac{1}{6}\right) \uplus \left(x^2 - x + \frac{1}{6}\right).$$

(3) For the monomial function $f(x) = x^n$, $P(f) = \frac{6n}{n^2+3n+2}x - \frac{2n-2}{n^2+3n+2}$ and $Q(f) = x^n - \frac{6n}{n^2+3n+2}x + \frac{2n-2}{n^2+3n+2}$. Hence

$$x^n = \left(\frac{6n}{n^2+3n+2}x - \frac{2n-2}{n^2+3n+2}\right) \uplus \left(x^n - \frac{6n}{n^2+3n+2}x + \frac{2n-2}{n^2+3n+2}\right).$$

(4) For $f(x) = \cos x$, $P(f) = (-12 + 6\sin 1 + 12\cos 1)\,x + 6 - 6\cos 1 - 2\sin 1$ and $Q(f) = (\cos x + (12 - 12\cos 1 - 6\sin 1)\,x - 6 + 6\cos 1 + 2\sin 1$. So that $f = P(f) \uplus Q(f)$.
(5) The last example is the natural exponential function $f(x) = e^x$, and its orthogonal decomposition is given as

$$e^x = \underbrace{(-6ex + 4e)}_{P(f)} \uplus \underbrace{(e^x + 6ex - 4e)}_{Q(f)}.$$

Indeed, let $\eta = D_0^{-4}(D^2 f) = D_0^{-4}(e^x)$ up on solving the differential equation with vanishing boundary conditions inversely, we have $\eta(x) = e^x + ex^3 - 2ex^2$. Consider $g := f - D^2\eta = -6ex + 4e$ so that $f = g \uplus D^2\eta$. Thus $e^x = (-6ex + 4e) \uplus (e^x + 6ex - 4e)$.
Up on calculations, we see $P^2(e^x) = P(-6ex + 4e) = -6ex + 4e$ and $(Q \circ P)(e^x) = Q(-6ex + 4e) = 0$ justifying the fact that $P^2 = P$ and $Q \circ P = 0$.

Proposition 3.4. *For $f \in W^{1,2}(\Omega)$, it holds that*

$$\langle Pf, Qf\rangle_{\mathcal{L}^2(\Omega)} = -\left\langle (Pf)', (Qf)'\right\rangle_{\mathcal{L}^2(\Omega)}.$$

Proof. This follows from the fact that Pf and Qf are orthogonal in the $W^{1,2}(\Omega)$. □

Example 3.5. (1) $\left\langle x - \frac{1}{6}, x^2 - x + \frac{1}{6}\right\rangle_{\mathcal{L}^2(\Omega)} = -\langle 1, 2x-1\rangle_{\mathcal{L}^2(\Omega)}$

(2) For $\alpha\beta = -1$, $\left\langle e^{\alpha x}, e^{\beta x}\right\rangle_{\mathcal{L}^2(\Omega)} = -\left\langle \alpha e^{\alpha x}, \beta e^{\beta x}\right\rangle_{\mathcal{L}^2(\Omega)}$

The space $W^{-1,2}(\Omega)$ of negative regularity exponent is the conjugate space of the Sobolev space $W^{1,2}(\Omega)$, i.e. $W^{-1,2}(\Omega) = (W^{1,2}(\Omega))^*$, where $(W^{1,2}(\Omega))^* := \{\tau : W^{1,2}(\Omega) \longrightarrow \mathbb{R}, \tau$ is a bounded linear functional$\}$.

We have $W_0^{1,2}(\Omega)^\perp = \left\{f \in W^{1,2}(\Omega) : \langle f, g\rangle = 0 \text{ for all } g \in W_0^{1,2}(\Omega)\right\}$ and from linear algebra of vector spaces we have the direct sum

$$W^{1,2}(\Omega) = W_0^{1,2}(\Omega) \oplus W_0^{1,2}(\Omega)^\perp$$

and therefore we have an interesting relation $W^{1,2}(\Omega) \ominus W_0^{1,2}(\Omega) = W_0^{1,2}(\Omega)^\perp$.

Theorem 3.6. *The direct difference or simply $W_0^{1,2}(\Omega)^\perp$ is a bi-codimensional subspace of $W^{1,2}(\Omega)$.*

Proof. Indeed $W_0^{1,2}(\Omega)^\perp = \left\{f \in W^{1,2}(\Omega) : \langle f, g\rangle = 0 \text{ for all } g \in W_0^{1,2}(\Omega)\right\}$, where $\langle f, g\rangle = \int_\Omega (fg + f'g')\, dx = 0$. Applying integration by parts with no boundary integrals as $g \in W_0^{1,2}(\Omega)$, we have

$\int_\Omega \left(fg - f''g\right) dx = \int_\Omega \left(f - f''\right) g dx = 0, \forall g \in W_0^{1,2}(\Omega) \Longrightarrow f - f'' = 0$

Solving the second order ordinary differential equation $f'' - f = 0$ we have $f_c = \alpha e^x + \beta e^{-x} \in W_0^{1,2}(\Omega)^\perp$ for α, β arbitrary real constants.

Therefore we have $\left(W_0^{1,2}(\Omega)\right)^\perp = \operatorname{span}\langle e^x, e^{-x}\rangle$. From the fact that $W^{1,2}(\Omega) = W_0^{1,2}(\Omega) \oplus \left(W_0^{1,2}(\Omega)\right)^\perp$ we have

$$W^{1,2}(\Omega) \ominus W_0^{1,2}(\Omega) = \left(W_0^{1,2}(\Omega)\right)^\perp = \operatorname{span}\langle e^x, e^{-x}\rangle.$$

Hence $W^{1,2}(\Omega) \ominus W_0^{1,2}(\Omega)$ is a skinny two dimensional subspace. This is interesting by itself, showing the fact that when we remove all elements that vanish on the boundary, the space what is left is a two dimensional subspace. □

From the direct sum $W^{1,2}(\Omega) = W_0^{1,2}(\Omega) \oplus \left(W_0^{1,2}(\Omega)\right)^\perp$ we have orthogonal projections $\widetilde{P} : W^{1,2}(\Omega) \longrightarrow W_0^{1,2}(\Omega)$ and $\widetilde{Q} : W^{1,2}(\Omega) = W_0^{1,2}(\Omega)^\perp$ such that $\widetilde{P}(f) = f - (\alpha e^x + \beta e^{-x})$, $\widetilde{Q}(f) = \alpha e^x + \beta e^{-x}$ for all $f \in W^{1,2}(\Omega)$, where $\alpha = \frac{f(1)e - f(0)}{e^2-1}$ and $\beta = \frac{f(0)e^2 - f(1)e}{e^2-1}$.

Proposition 3.7. $\widetilde{P}(f)_{|\partial\Omega} = 0$ *and* $\widetilde{Q}f_{|\partial\Omega} = f_{|\partial\Omega}$.

Proof. Clearly $\widetilde{P}(f)(0) = 0$ and $\widetilde{P}(f)(1) = 0$

$\widetilde{Q}(f)(0) = \alpha + \beta = \frac{f(1)e - f(0)}{e^2-1} + \frac{f(0)e^2 - f(1)e}{e^2-1} = \frac{f(0)(e^2-1)}{e^2-1} = f(0)$ and

$\widetilde{Q}(f)(1) = \alpha e + \beta e^{-1} = \frac{f(1)e^2 - f(0)e}{e^2-1} + \frac{f(0)e - f(1)}{e^2-1} = \frac{f(1)\left(e^2-1\right)}{e^2-1} = f(1)$. □

Now from $W^{1,2}(\Omega) = W_0^{1,2}(\Omega) \oplus \left(W_0^{1,2}(\Omega)\right)^{\perp}$ we have

$$W^{1,2}(\Omega)^* = W_0^{1,2}(\Omega)^* \oplus \left(W_0^{1,2}(\Omega)\right)^{\perp *}.$$

That is

$$W^{-1,2}(\Omega) = W_0^{-1,2}(\Omega)^* \oplus \left(W_0^{1,2}(\Omega)\right)^{\perp *}.$$

Looking $W_0^{-1,2}(\Omega)^* = \left\{\zeta : W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}, \text{ bounded linear functional}\right\}$and from the Riesz representation theorem there exists $f \in W_0^{1,2}(\Omega)$ such that $\gamma(g) = \langle \gamma, g \rangle = \int_\Omega gf + g'f'dx$ for all $g \in W_0^{1,2}(\Omega)$.

Then we have the representation of γ to be $\gamma = f - f''$, where $f,\ f' \in \mathcal{L}^2(\Omega)$.

If $\gamma \neq 0$, then the function f that is used to represent γ is a solution of the inhomogeneous differential equation $\gamma = f - f''$.

Special interest: The function that represents the zero linear functional $\gamma = 0$ is a solution of the homogeneous ordinary differential equation given by $f'' - f = 0$.

Up on solving the latter second order ordinary differential equation, we get the function that represents the zero linear functional to be

$$f_c = \alpha e^x + \beta e^{-x} \tag{3.2}$$

with α, β some real constants.

Proposition 3.8 (Representation of a linear functional)**.** *Let $\zeta : W_0^{1,2}(\Omega) \longrightarrow \mathbb{R}$ be a bounded linear functional, then by the Riesz representation theorem, $\exists f \in W_0^{1,2}(\Omega)$ such that $\zeta = f_0 - \frac{d}{dx} f_1$ with $f_0 = f, \quad f_1 = \frac{d}{dx} f = f'$, where f, $f_1 \in \mathcal{L}^2(\Omega)$.*

Proof. For $\zeta \in W_0^{1,2}(\Omega)^*$, $\exists f \in W_0^{1,2}(\Omega) : \zeta(g) = \langle \zeta, g \rangle = \int_\Omega (fg + f'g')\, dx, \forall g \in W_0^{1,2}(\Omega)$. But

$$\int_\Omega (fg + f'g')\, dx = \int_\Omega \left(fg - f''g\right) dx = \int_\Omega (f - f'')\, g dx = \langle f - f'', g \rangle.$$

Therefore $\zeta = f_0 - \frac{d}{dx} f_1$ with $f_0 = f,\ f_1 = \frac{d}{dx} f = f'$, where $f_0,\ f_1 \in \mathcal{L}^2(\Omega)$. ☐

Definition 3.9. Let $f, g : \Omega \longrightarrow \mathbb{R}$ with $\| g \|_{W^{1,2}(\Omega)} \neq 0$. Then we define the projection of f over g by $\text{Proj}_g(f)_{W^{1,2}(\Omega)} := \frac{\langle f,g \rangle_{W^{1,2}(\Omega)}}{\|g\|^2_{W^{1,2}(\Omega)}} g$.

The last result reads as follows. Its proof is easy and so we omit it.

Proposition 3.10. *For $f, g, h : \Omega \longrightarrow \mathbb{R}$ with $\| h \|_{W^{1,2}(\Omega)} \neq 0$ and $\alpha \in \mathbb{R}$, the following statements hold:*

(1) $\text{Proj}_h(f+g)_{W^{1,2}(\Omega)} = \text{Pr}oj_h(f)_{W^{1,2}(\Omega)} + \text{Proj}_h(g)_{W^{1,2}(\Omega)}$.
(2) $\text{Proj}_h(\alpha f)_{W^{1,2}(\Omega)} = \alpha \text{Proj}_h(f)_{W^{1,2}(\Omega)}$.
(3) *If the two functions are orthogonal, then* $\text{Proj}_g(f)_{W^{1,2}(\Omega)} = 0$.
(4) $\text{Proj}_f(f)_{W^{1,2}(\Omega)} = f$.
(5) $\text{Proj}_{(\beta g)}(f)_{W^{1,2}(\Omega)} = \text{Proj}_g(f)_{W^{1,2}(\Omega)}$.
(6) $\text{Proj}_{(\beta g)}(\alpha f)_{W^{1,2}(\Omega)} = \alpha \text{Proj}_g(f)_{W^{1,2}(\Omega)}$.
(7) $\text{Proj}_{(x^2)} e^x_{W^{1,2}(\Omega)} = \frac{15e}{23} x^2$.

(8) $\text{Proj}\,_{(e^{-x})}\, e^x_{W^{1,2}(\Omega)} = 0.$

(9) $\text{Proj}\,_{(\cos x)}\, (\sin x)_{W^{1,2}(\Omega)} = 0.$

(10) $\text{Proj}\,_{(e^{\beta x})}\, e^{\alpha x}_{W^{1,2}(\Omega)} = \gamma e^{\beta x}$ *for* $\alpha, \beta \neq 0$ *and* $\gamma = \frac{(2\alpha+2\beta)e^{\alpha+\beta} - 2\alpha\beta^2 - 2\beta}{(\alpha+\beta)(\beta^2+1)\left(e^{2\beta}-1\right)}$.

Acknowledgements. The author would like to thank the anonymous reviewer, who read the article and propose some positive amendments on the format of the article.

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