The Fibonacci Like Numbers

By

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In this short note, we indeed generate sequences of numbers not necessarily integers that are Fibonacci like through the *discrete Laplace transform* method (DLT). In the process, the Fibonacci numbers we know will be particular case of the general sequence we obtain.

The Fibonacci numbers are one of the wonders of old mathematics. They represent several natural things, such as leaves of trees, foliages of flowers, replications of some species, etc. These Fibonacci numbers are given by the sequence :

$$1, 1, 2, 3, 5, 8, 13, 21, 34, ...$$

The pattern is that a number is the sum of two of the previous or predecessor numbers and can be written as :

$$a_{n+2} = a_{n+1} + a_n \text{ for } n = 1, 2, 3, \text{....with } a_1 = 1, a_2 = 1$$

We know that such a sequence has a formula that generates these numbers. That formula is :

$$a_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}2^n}$$

In discrete mathematics or sequence courses we ask students to verify using mathematical induction that indeed, the given formula generates the Fibonacci numbers. It is also a known fact that the quotients of consecutive numbers of the sequence converges to a number:

$$\frac{a_{n+1}}{a_n} \longrightarrow \frac{1+\sqrt{5}}{2} \text{ as } n \to \infty$$

We use a powerful method: the *discrete Laplace transform*, (for more reading on the transform, see www.dejeniea.com/disclaptrans_Addis_.pdf) that generates solutions to many problems that are described in terms of sequences, or difference equations which are prevalent in discrete mathematics. We will see how the method is used to find solutions of equations such as:

$$a_{n+1} = \lambda a_n + \beta, a_1 = a(1),\, n = 1, 2, 3, ..$$

$$a_{n+2} = a_{n+1} + a_n, a_1 = a\left(1\right), a_2 = a\left(2\right),\, n = 1, 2, 3, ...$$

Let

$$f : \mathbb{N} \rightarrow \mathbb{R}$$

be a sequence and let $s > 0$. We define the discrete Laplace transform of f by:

$$\ell_d \{f(n)\}(s) := \sum_{n=1}^{\infty} f(n) e^{-sn}$$

provided the series converges.

Existence of a discrete Laplace Transform:

Let $f(n) : \mathbb{N} \rightarrow \mathbb{R}$ be a sequence such that

$$| f(n) | \leq \alpha e^{s_0 n} \text{ for } \alpha > 0, s_0 > 0.$$

Then

$$\sum_{n=1}^{\infty} f(n) e^{-sn}$$

is absolutely convergent and hence is convergent. Therefore, for such a sequence, the discrete Laplace transform

$$\ell_d \{f(n)\}(s)$$

exists finitely for $s > s_0$, since

$$| \sum_{n=1}^{\infty} f(n) e^{-sn} | \leq \sum_{n=1}^{\infty} \alpha e^{(s_0 - s)n} = \frac{\alpha}{e^{s-s_0} - 1} < +\infty$$

for $s > s_0$. From this, we conclude that sequences which are polynomials in n have discrete Laplace transform.

Proposition 1

$$\ell_d \left\{a^{n-1}\right\}(s) = \frac{1}{e^s - a}$$

provided $0 < a < e^s$.

Proof. From the definition,

$$\ell_d \left\{a^{n-1}\right\}(s) = \sum_{n=1}^{\infty} e^{-sn} a^{n-1}$$

$$= \sum_{n=1}^{\infty} e^{-sn} e^{(n-1)\ln a}$$

$$= a^{-1} \sum_{n=1}^{\infty} e^{-sn} e^{n \ln a}$$

$$= a^{-1} \sum_{n=1}^{\infty} e^{-(s-\ln a)n}$$
$$= \frac{1}{e^s - a}$$

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Example 2

$$\ell_d \left\{5^{n-1}\right\}(s) = \frac{1}{e^s - 5}$$

Proposition 3 *The numbers:*

$$1, 1, 2, 3, 5, 8, 13, ...$$

usually called the Fibonacci sequence are generated by the formula :

$$a_n = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}2^n}$$

Proof. First, we all know, the numbers can be represented in a recursive way by :

$$a_{n+2} = a_{n+1} + a_n \text{ for } n = 1, 2, 3, ...\text{with } a_1 = 1, a_2 = 1.$$

We will see in a moment that this sequence as a special case of a general sequence that will be generated from the one whose formula will be obtained from the recursive expression :

$$a_{n+2} = a_{n+1} + a_n \text{ for } n = 1, 2, 3, ...\text{with } a_1 = a(1), a_2 = a(2)$$

as two initial conditions. The pattern that will be observed from the latter sequence is that the general term of the sequence will appear as a linear combination of the two initial conditions a_1 and a_2:

$$a_n = \gamma_n a_1 + \beta_n a_2$$

in which γ_n, β_n are themselves obtained as Fibonacci numbers of the respective coefficients of a_1 and a_2. I call these numbers Fibonacci - like numbers.

Applying the discrete Laplace transform on both sides of the later recursive equation :

$$\ell_d\{a_{n+2}\}(s) = \ell_d\{a_{n+1} + a_n\}(s) = \ell_d\{a_{n+1}\}(s) + \ell_d\{a_n\}(s).$$

From results of Dejenie A. Lakew (2010) we have transforms of these types:

$$
\begin{aligned}
\ell_d\{a_{n+2}\}(s) &= e^{2s}\ell_d\{a_n\}(s) - e^s a_1 - a_2 \\
\ell_d\{a_{n+1}\}(s) + \ell_d\{a_n\}(s) &= e^s \ell_d\{a_n\}(s) - a_1 + \ell_d\{a_n\}(s)
\end{aligned}
$$

$\Rightarrow$

$$
e^{2s}\ell_d\{a_n\}(s) - e^s a_1 - a_2 = e^s\ell_d\{a_n\}(s) - a_1 + \ell_d\{a_n\}(s)
$$

Therefore rearranging, we have :

$$
\ell_d\{a_n\}(s) = \frac{a_1 e^s + a_2 - a_1}{e^{2s} - e^s - 1}
$$

But

$$
e^{2s} - e^s - 1 = \left(e^s - \frac{(1+\sqrt{5})}{2}\right)\left(e^s - \frac{(1-\sqrt{5})}{2}\right)
$$

and

$$
\begin{aligned}
\frac{a_1 e^s + a_2 - a_1}{e^{2s} - e^s - 1} &= \frac{a_1 e^s + a_2 - a_1}{\left(e^s - \frac{(1+\sqrt{5})}{2}\right)\left(e^s - \frac{(1-\sqrt{5})}{2}\right)} \\
&= \frac{a_1\left(\sqrt{5}-1\right) + 2a_2}{2\sqrt{5}\left(e^s - \frac{(1+\sqrt{5})}{2}\right)} + \frac{a_1\left(\sqrt{5}+1\right) - 2a_2}{2\sqrt{5}\left(e^s - \frac{(1-\sqrt{5})}{2}\right)}
\end{aligned}
$$

Therefore taking the inverse discrete Laplace transform we have :

$$
a_n = \ell_d^{-1}\left\{\frac{a_1\left(\sqrt{5}-1\right) + 2a_2}{2\sqrt{5}\left(e^s - \frac{(1+\sqrt{5})}{2}\right)} + \frac{a_1\left(\sqrt{5}+1\right) - 2a_2}{2\sqrt{5}\left(e^s - \frac{(1-\sqrt{5})}{2}\right)}\right\}
$$

$$
= \frac{1}{\sqrt{5}2^n}\left(\left(a_1\left(\sqrt{5}-1\right) + 2a_2\right)\left(1+\sqrt{5}\right)^{n-1} + \left(a_1\left(\sqrt{5}+1\right) - 2a_2\right)\left(1-\sqrt{5}\right)^{n-1}\right)
$$

Therefore, the sequence

$$
a_n = \frac{1}{\sqrt{5}2^n}\left(\left(a_1\left(\sqrt{5}-1\right) + 2a_2\right)\left(1+\sqrt{5}\right)^{n-1} + \left(a_1\left(\sqrt{5}+1\right) - 2a_2\right)\left(1-\sqrt{5}\right)^{n-1}\right)
$$

is a solution to the recursive equation of the Fibonacci-like numbers. Rearranging the expression, we get linear combinations of a_1 and a_2 as :

$$a_n = \frac{\left(\left(\sqrt{5}-1\right)\left(1+\sqrt{5}\right)^{n-1}+\left(\sqrt{5}+1\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)}{\sqrt{5}2^n}a_1+\frac{\left(\left(1+\sqrt{5}\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)}{\sqrt{5}2^{n-1}}a_2$$

with coefficients of a_1 and a_2 being represented by:

$$\gamma_n = \frac{\left(\left(\sqrt{5}-1\right)\left(1+\sqrt{5}\right)^{n-1}+\left(\sqrt{5}+1\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)}{\sqrt{5}2^n}$$

and

$$\beta_n = \frac{\left(\left(1+\sqrt{5}\right)^{n-1}-\left(1-\sqrt{5}\right)^{n-1}\right)}{\sqrt{5}2^{n-1}}$$

We note that when $n = 1$, the coefficient of a_1 is 1 and that of a_2 is zero and therefore the term will be just a_1. Like wise when $n = 2$, the coefficient of a_1 is zero and that of a_2 is one and again the term will be a_2. The coefficients themselves are generated as a sequence which are Fibonacci like numbers.

Then coming back to our original question, extracting a sequence that generates the Fibonacci sequence, we look at the above general sequence but with two fixed initial conditions :

$$a_1 = 1 = a_2$$

and that we get the following sequence which generates the well known Fibonacci numbers that are known to be integers:

$$a_n = \frac{\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n}{\sqrt{5}2^n}, n \in \mathbb{N}$$

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The other sequences for initial values other than 1 will generate numbers which are not necessarily integers but ruled by the sequential defintion indicated at the beginning.
It will be a challenge to find which initial conditions will result in with a sequence of integer values. It is a problem that somebody can persue.

Proposition 4 *Let $\lambda\,(\neq 1) \in \mathbb{R}$, the solution to the recursive equation:*

$$a_{n+1} = \lambda a_n + \beta, a\left(1\right) = a_1, n \in \mathbb{N}$$

is given by

$$a_n = \left(a_1 + \frac{\beta}{\lambda - 1}\right)\lambda^{n-1} + \frac{\beta}{1-\lambda}, n \in \mathbb{N}$$

Proof. Using the discrete replace transform of both sides of the equation in the proposition, we have:

$$\begin{aligned} l_d\{a_n\} &= \frac{a_1}{e^s-\lambda}+\frac{\beta}{(e^s-1)(e^s-\lambda)} \\ &= \frac{a_1}{e^s-\lambda}+\frac{\beta}{(\lambda-1)(e^s-1)}+\frac{\beta}{(1-\lambda)(e^s-1)} \\ &= \frac{\left(a_1+\frac{\beta}{\lambda-1}\right)}{e^s-\lambda}+\frac{\beta}{(1-\lambda)(e^s-1)} \end{aligned}$$

Then taking the inverse discrete Laplace transform, we have:

$$a_n=\left(a_1+\frac{\beta}{\lambda-1}\right)\lambda^{n-1}+\frac{\beta}{1-\lambda}, n\in\mathbb{N}$$

Note here why we restrict $\lambda\neq 1$.

The case for $\lambda=1$ is done in the following way: considering the equation :

$$a_{n+1}=a_n+\beta, n=1,2,3,...$$

and taking the discrete Laplace transform of both sides, and solving for $l_d\{a_n\}$ we get

$$l_d\{a_n\}=\frac{a_1}{e^s-1}+\frac{\beta}{(e^s-1)^2}$$

Applying the inverse discrete Laplace transform we have,

$$\begin{aligned} a_n &= l_d^{-1}\left\{\frac{a_1}{e^s-1}\right\}+l_d^{-1}\left\{\frac{\beta}{(e^s-1)^2}\right\} \\ &= a_1 l_d^{-1}\left\{\frac{1}{e^s-1}\right\}+\beta l_d^{-1}\left\{\frac{1}{(e^s-1)^2}\right\} \\ &= a_1+\beta(1*1) \\ &= a_1+\beta(n-1) \end{aligned}$$

where $1*1$ is the convolution of the constant sequence 1 by itself, which is $n-1$. Therefore this case has a solution given by :

$$a_n=\beta(n-1)+a_1, n\in\mathbb{N}$$

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