MOLLIFIERS IN CLIFFORD ANALYSIS

DEJENIE A. LAKEW

Dedicated to the Memory of our Dear Mother

ABSTRACT. We introduce mollifiers in Clifford analysis setting and construct a sequence of $C^\infty-$functions that approximates a $\gamma-$regular function and also a solution to a non homogeneous BVP of an in-homogeneous Dirac like operator in certain Sobolev spaces over bounded domains whose boundary is not that wild. One can extend the smooth functions upto the boundary if the domain has a C^1- boundary and this is the case in the paper as we consider a domain whose boundary is a C^2-hyper surface.

1. Introduction: Algebraic and Analytic Rudiments

Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary is a C^2-hyper surface and Cl_n be a 2^n- dimensional Clifford algebra generated by $\mathbb{R}^n$ with an inner product that satisfies $x^2 = -\parallel x \parallel^2$.

Then for $e_1, e_2, ..., e_n$ which are orthonormal basis of $\mathbb{R}^n$, we have an equality $e_{ij} + e_{ji} = -\delta_{ij}e_0$, with δ_{ij} , the Kronecker delta symbol and e_0, the identity element of the Clifford algebra.

A Cl_n-valued function f defined in Ω has a standard representation :

$$f(x) = \sum_A e_A f_A(x), x \in \Omega \tag{1.1}$$

where for each index set A, $f_A : \Omega \to \mathbb{R}$ is a real valued section of f.

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Such a function f is continuous, differentiable, integrable, measurable, etc. over Ω, if each section f_A of f is respectively continuous, differentiable, integrable, measurable, etc. over Ω.

Thus the usual function spaces, the Hölder spaces denoted by $C^{\alpha}(\Omega, Cl_n), C^{m,\alpha}(\Omega, Cl_n)$ and the Sobolev spaces denoted by $W^{p,k}(\Omega, Cl_n)$ for $m, k = 0, 1, ...$ and $1 < p < \infty$, are defined as follows:

$f \in C^{\alpha}(\Omega, Cl_n)$ iff $f_A \in C^{\alpha}(\Omega, \mathbb{R})$ where $C^{\alpha}(\Omega, \mathbb{R})$ is the space of all functions f which are Hölder continuous with Hölder exponent α :

$$\mid f(x) - f(y) \mid \leq k_f \mid x - y \mid^{\alpha}$$

for $x, y \in \Omega$ with norm given by :

$$\|f\|_{C^{\alpha}(\Omega,\mathbb{R})} = \| f \|_{C(\Omega,\mathbb{R})} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\mid f(x) - f(y) \mid}{\mid x - y \mid^{\alpha}}$$

where k_f is a positive constant which is specific to the particular function f .

For a very trivial fact, the Hölder exponent α should be in the interval $(0, 1]$, for otherwise, if $\alpha > 1$, we have

$$\frac{\mid f(x) - f(y) \mid}{\mid x - y \mid} \leq k_f \mid x - y \mid^{\zeta}$$

for some $\zeta > 0$ and some positive constant k_f .

When $\alpha = 1$, the functions are called Lipschitz functions and these functions have bounded derivatives over the domain Ω.

Also, $f \in C^{m,\alpha}(\Omega, Cl_n)$ iff $f_A \in C^{m,\alpha}(\Omega, \mathbb{R})$ where $C^{m,\alpha}(\Omega, \mathbb{R})$ is the space of functions $f : \Omega \to \mathbb{R}$ which are $m-$times continuously differentiable and whose $m-$th derivative is Hölder continuous with exponent α and with norm given by

$$\begin{aligned} \| \quad f \|_{C^{m,\alpha}(\Omega,\mathbb{R})} &= \| f \|_{C^{m-1}(\Omega,\mathbb{R})} + \| f^{(m)} \|_{C^{\alpha}(\Omega,\mathbb{R})} \\ &= \| f \|_{C^{m}(\Omega,\mathbb{R})} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{\mid f^{(m)}(x) - f^{(m)}(y) \mid}{\mid x - y \mid^{\alpha}} \end{aligned}$$

Finally for $p \in [1, \infty)$, Sobolev spaces are defined in a similar way:

$f \in W^{p,k}(\Omega, Cl_n)$ iff $f_A \in W^{p,k}(\Omega, \mathbb{R})$ where $W^{p,k}(\Omega, \mathbb{R})$ is the space of real valued functions f defined over Ω which are locally $p-$ integrable over Ω and whose $j-th$ distributional (or weak) derivatives $D^j f$ with $\mid j \mid \leq k$ exist and are all $p-$integrable over Ω and norm in such a space is defined as :

$$\parallel f \parallel_{W^{p,k}(\Omega,\mathbb{R})} = \left(\sum_{|j| \leq k} \parallel D^j f \parallel^p_{L^p(\Omega,\mathbb{R})} \right)^{p^{-1}}$$

Here, a locally integrable function $f : \Omega \to \mathbb{R}$ is said to have a locally integrable $j-th$ order distributional (or weak) derivative over Ω if and only if

$$\int_\Omega D^j f(x)\, \phi(x)\, d\Omega_x = (-1)^{|j|} \int_\Omega f(x)\, D^j \phi(x)$$

for all test functions $\phi \in C_c^\infty(\Omega)$, and $D^j = \prod_{i=1}^{n} \frac{\partial^{j_i}}{\partial x_i^{j_i}}$ with j a multi-index exponent such that $\sum_{i=1}^{n} j_i = j$.

Note here that $W^{p,0}(\Omega, Cl_n) = L^p(\Omega, Cl_n)$, the Lebesgue space of $p-$integrable Clifford valued functions and for a detail study of function spaces, one can refer [7, 8, 9, 13]

For $p = 2$, the Lebesgue space $L^2(\Omega, Cl_n)$ becomes a Hilbert space with a Clifford-valued inner product given by

$$\langle f, g \rangle_\Omega := \int_\Omega \overline{f(x)} g(x) d\Omega \tag{1.2}$$

Introduce the in-homogeneous **Dirac**-operator with gradient potential γ by:

$$D_\gamma := \sum_{j=1}^{n} e_j \left(\frac{\partial}{\partial x_j} - \gamma_j \right) \tag{1.3}$$

where $\gamma = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} \Gamma$ (with $\Gamma \in C^1(\Omega \to \mathbb{R})$ linear) is called the *gradient potential* of Γ

Definition 1. *A function $f \in C^1(\Omega \to Cl_n)$ is said to be left γ-**regular** if $D_\gamma f(x) = 0$, $\forall x \in \Omega$ and right γ-**regular** if $f(x)D_\gamma = 0$.*

An example of a function which is both left and right $\gamma-$regular over Ω is given by

$$\Psi^\Gamma(x) := \frac{\overline{x}}{\omega_n \|x\|^n} e^{-\Gamma(x)} \tag{1.4}$$

where $\omega_n = \frac{\sqrt{\pi^n}}{\Gamma\left(\frac{n+2}{2}\right)}$ is the surface area of the unit sphere in $\mathbb{R}^n$.

The function given above is also called a **fundamental solution** (or **Cauchy kernel**) for the in-homogeneous Dirac operator D_γ

Proposition 1. *Let Ω be a bounded, C^2-domain in $\mathbb{R}^n$ and let $g \in W_\Gamma^{2,k-\frac{1}{2}}(\partial\Omega, Cl_n)$, $k = 1, 2, ...$ Then the*

$$\mathbf{BVP}: \quad \begin{cases} D_\gamma f = 0 & \text{on } \Omega \\ tr f = g & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

has a solution $f \in W_\Gamma^{2,k}(\Omega, Cl_n)$ given by

$$f(x) = \int_{\partial\Omega} \Psi_\Gamma(x-y)\nu(y)g(y)d\Sigma_y, \ x \in \Omega \tag{1.6}$$

The theme here is to construct Clifford valued $C^\infty-$ function g over Ω that approximates the solution function f in the $H^k(\Omega, Cl_n)$ $\left(\text{or } W^{2,k}(\Omega, Cl_n)\right)$ sense and also to approximate the solution of a non homogeneous boundary value problem on Sobolev spaces based at $L^p(\Omega, Cl_n)$:

$$\mathbf{NHBVP}: \begin{cases} D_\gamma f = h & \text{on } \Omega \\ tr f = g & \text{on } \partial\Omega \end{cases}$$

whose solution is given by : $W^{p,k}(\Omega, Cl_n) \ni f = \Psi^\Gamma *_{|\partial\Omega} \nu\left(tr_{\partial\Omega} f\right) + \Psi^\Gamma *_{|\Omega} (D_\gamma f)$ and substituting $tr_\Omega f = g$ and $D_\gamma f = h$ on Ω, where $g \in W^{p,k-\frac{1}{p}}(\partial\Omega, Cl_n)$ and $h \in W^{p,k-1}(\Omega, Cl_n)$ where the result is given in **Proposition** 6.

This is possible by constructing a smooth function g over any sub domain $\Delta \subset\subset \Omega$, where for each $\delta > 0$, we have that

$$\|f - g\|_{W_\Gamma^{2,k}(\Delta, Cl_n)} < \delta$$

and taking the supremum of such approximations over all such possible sub domains as

$$\sup_{\Delta\subset\subset\Omega} \|f-g\|_{W^{2,k}_{\Gamma}(\Delta,Cl_n)}$$

we get the result.

The smooth functions in general are constructed using mollifiers which sooth locally or globally integrable functions in certain Sobolev spaces and the notation $\Delta \subset\subset \Omega$ read as "Δ is compactly contained in Ω " is to mean that Δ is a subset of Ω whose compact closure $\overline{\Delta}$ is also contained in Ω.

In [6] , the author constructed a family of functions which are called minimal to approximate in the best way, such a $\gamma-$regular function with finitely many of these functions. For detail results, see the reference therein.

2. **Approximations with Smooth Functions**

As I mentioned above, in [6] we construct Cl_n-minimal family of functions in $B^2_{\gamma}(\Omega, Cl_n)$ which are used for approximating solutions of elliptic boundary value problems in the best way. The construction was made by choosing dense points of some outer surface and define a family of functions from the fundamental solution Ψ^{Γ} of the in-homogeneous Dirac operator D_{γ} with the selected points as the singular points of the fundamental solution. We then refine these functions more by an orthogonalization like process. The approximating functions constructed in this way were in the Sobolev space where the function to be approximated belongs.

But what we intend to do here is that the same function which is approximated by minimal family of functions can also be approximated by smooth functions(in fact $C^{\infty}-$functions) over the domain Ω.

We shall mention that the smooth approximation over the domain is always possible as long as the function is integrable over the domain, and this approximation is extendable up to the boundary if the boundary of the domain is a C^1- hypersurface. Therefore, when the domain is Lipschitz(minimal smoothness condition on the boundary),

the approximating smooth functions may not be extendable up to the boundary.

We therefore start with the notion of a mollifier. As a Cl_n-valued function f has a general representation given by (1.1), we start with mollifying a real valued function and then we extend that definition to that of a Clifford valued function.

Let Ω be a bounded domain with a C^1-boundary, and for ϵ be a positive constant, define a sub domain Ω_ϵ of Ω by $\Omega_\epsilon := \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > \epsilon\}$.

Let us also consider the function

$$\phi\,(x) = \chi_{\overset{0}{B(0,1)}}\, k e^{\left(\|x\|^2-1\right)^{-1}} \tag{2.1}$$

which is a $C^\infty-$function over $\mathbb{R}^n$ whose compact support is within the unit ball $B\,(0,1)$ and we choose the constant k so that the integral of ϕ over the space $\mathbb{R}^n$ is a unit. The function $\chi_{\overset{0}{B}}$ is the characteristic function of the interior of the unit ball $B\,(0,1)$.

Then for a function $f : \Omega \to \mathbb{R}$ which is locally integrable, we define the convolution :

$$f^\epsilon\,(x) := \int\limits_\Omega \epsilon^{-n}\phi\left(\frac{x-y}{\epsilon}\right) f\,(y)\, d\Omega_y \tag{2.2}$$

which is the convolution of the mollifier function ϕ_ϵ with that of f over the sub domain Ω_ϵ where $\phi_\epsilon\,(x) = \epsilon^{-n}\phi\,(\epsilon^{-1}x)$ is a $C^\infty-$function compactly supported in the $\epsilon-$ball centered at the origin. The above function f^ϵ defined as $f^\epsilon := \phi_\epsilon * f$ is some times called a regularization of f.

Lemma 1. *The convolution function f^ϵ is a $C^\infty-$function over the $\epsilon-$thick skin removed sub domain Ω_ϵ and besides $\lim\limits_{\epsilon\downarrow 0} f^\epsilon = f$ in measure.*

Proposition 2. *(Clifford Analysis version of a regularization)*

For $f = \sum_A e_A f_A : \Omega \to Cl_n$ and $f_A^\epsilon := \phi_\epsilon * f_A$, the regularization $f^\epsilon = \sum\limits_A e_A f_A^\epsilon$ is $C^\infty-$over Ω_ϵ and further more $\lim\limits_{\epsilon\longrightarrow 0}\left(\sum\limits_A e_A f_A^\epsilon\right) = f$

Proof. For each index set A, f_A is a real valued function from the domain Ω and by the above lemma, the convolution $f_A^\epsilon = \phi_\epsilon * f_A$ is a $C^\infty-$function over Ω_ϵ. Then the Clifford sum of such smooth functions

: $\sum_A e_A (\phi_\epsilon * f_A) =: f^\epsilon$ is a smooth function as well. Also by continuity, $\sum_A e_A (\phi_\epsilon * f_A) \to \sum_A e_A f_A$ as $\epsilon \to 0$, that is $f^\epsilon \to f$ as $\epsilon \to 0$. □

The following proposition is the main result of the paper.

Proposition 3. *Let Ω be a bounded domain with a C^1- boundary and $1 < p < \infty$, $f \in W^{p,k}(\Omega, Cl_n)$ where, $k = 0, 1, 2, ...$ Then $\forall \epsilon > 0$, there exists a Cl_n- valued function $\Psi = \sum_A e_A \psi_A$ over Ω which is $C^\infty-$ up to the boundary such that*

$$\| f - \Psi \|_{W^{p,k}(\Omega \cup \partial\Omega, Cl_n)} < \epsilon$$

Proof. We first start with the Clifford Analysis version of regularization .

For a Cl_n-valued function f defined on Ω which is represented by $f(x) = \sum_A e_A f_A(x)$, we construct locally integrable $C^\infty-$functions from f as

$$f^\epsilon := \sum_A e_A f_A^\epsilon \tag{2.3}$$

where, for each A,

$$f_A^\epsilon = \phi_\epsilon * f_A$$

From the construction of the mollifiers ϕ_ϵ, one can show that the $\epsilon-$wide section f^ϵ of the Clifford valued function f is $C^\infty-$ function over the sub domain Ω_ϵ as each component function f_A^ϵ is $C^\infty-$ over Ω_ϵ and

$$\lim_{\epsilon \downarrow 0} \left(\sum_A e_A (f_A * \phi_\epsilon) \right) = f = \sum_A e_A f_A$$

in measure over Ω.

The next procedure is to look at how each $\mathbb{R}-$valued component function f_A of the Cl_n-valued function f is approximated by $C^\infty-$functions (for more information on this particular procedure, one can refer [9]).

The process is out lined next, where some kind of *surgery* on the domain Ω is performed in order to construct smooth functions that will approximate f_A in terms of other smooth functions called partitions of unity (refer [9] for details) and then we extend the result to work for a Cl_n-valued function f .

To explain exactly what is happening is that we cut off each component function which is in a Sobolev space that may have a singularity of some order, by $C^{\infty}-$ functions which control the singularity and sooth the function and then we patch the smooth sections to create the needed $C^{\infty}-$ approximating functions.

Thus, for each $i, (i = 1, 2, ..,)$ construct sub domain $\Omega_i := \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > i^{-1}\}$ so that $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ and also consider the decomposition of the domain in the following way: $\widetilde{\Omega}_i = \Omega_{i+3} - \overline{\Omega}_{i+1}$, and then pick a sub domain $\widetilde{\Omega}_0 \subset\subset \Omega$ so that $\Omega = \bigcup_{i=0}^{\infty} \widetilde{\Omega}_i$. Then for an $\mathbb{R}-$valued component function $f_A \in W^{p,k}(\Omega)$ of the Clifford valued function f and for any partition of unity $\{\theta_i\}_{i=0}^{\infty}$ associated to the open cover $\left\{\widetilde{\Omega}_i\right\}_{i=0}^{\infty}$ of Ω, the function $\theta_i f_A$ is compactly supported over $\widetilde{\Omega}_i$ and furthermore, it is in the Sobolev space $W^{p,k}(\Omega, \mathbb{R})$.

Let us consider a $\beta > 0$ and choose a positive but small number ϵ_i such that the convolution function $\phi_{\in_i} * (\theta_i f_A) =: g_i$ has a compact support in $V_i := \Omega_{i+4} - \overline{\Omega}_i$ which contains $\widetilde{\Omega}_i$ for $i, (i = 1, 2, ...)$, and that satisfies the inequality:

$$\| g_i - \theta_i f_A \|_{W^{p,k}(\Omega,\mathbb{R})} \leq \frac{\beta}{2^{i+1}}$$

for $i = 0, 1, 2, ...$

Now let us consider the function $\psi := \sum_{i=0}^{\infty} g_i$ and we claim that ψ is a C^{∞}-function over Ω.

Indeed, for any open sub domain $\Delta \subset\subset \Omega$, we have $\psi_m := \psi_{|\Delta} = \sum_{i=0}^{m} g_i$ for some $m \in \mathbb{N}$, since $\Delta \subset\subset \Omega$, we have that $\overline{\Delta}^{\text{cpct}} \subsetneq \Omega$ so that finitely many of the sets from the cover $\{V_i\}_i$ of Ω covers Δ. Therefore, for any set $\Delta \subset\subset \Omega$ and for a section f_A of f we have the inequality

$$\| \psi_{|\Delta} - (f_A)_{|\Delta} \|_{|W^{p,k}(\Omega,\mathbb{R})} = \| \left(\sum_{i=0}^{\infty} g_i - \sum_{i=0}^{\infty} \theta_i f_A \right)_{|\Delta} \|_{|W^{p,k}(\Omega,\mathbb{R})}$$

$$= \| \underbrace{\sum_{i=0}^{\infty} (g_i - (\theta_i f_A))}_{\text{finite sum as } \Delta \subset\subset \Omega} \|_{|W^{p,k}(\Delta,\mathbb{R})}$$

$$\leq \sum_{i=0}^{m} \| g_i - (\theta_i f_A) \|_{|W^{p,k}(\Omega,\mathbb{R})} \leq \sum_{i=0}^{\infty} \frac{\beta}{2^{i+1}} = \beta$$

where f_A is represented by $f_A = \sum_{i=0}^{\infty} \theta_i f_A$.

Therefore, considering the $\sup_{\Delta \subset \Omega} \| \psi - f_A \|_{|W^{p,k}(\Delta,\mathbb{R})}$ we have the required result

$$\| \psi - f_A \|_{|W^{p,k}(\Omega \cup \partial\Omega,\mathbb{R})} \leq \beta$$

That is, the smooth function $\psi \left(= \lim_{m\to\infty} \psi_m = \lim_{m\to\infty} \left(\psi_{|\Delta}\right)\right)$ approximates f_A in the Sobolev space $W^{p,k}(\Omega \cup \partial\Omega, \mathbb{R})$.

Then since each $\mathbb{R}-$ valued component function f_A of the Cl_n-valued function $f = \sum_A e_A f_A$ is smoothly approximated over $\overline{\Omega} (= \Omega \cup \partial\Omega)$ by $\psi_A \in C^{\infty}(\Omega \cup \partial\Omega, \mathbb{R})$, we have that $\Psi = \sum_A e_A \psi_A$ approximates the whole function f over $\Omega \cup \partial\Omega$ which is $\overline{\Omega}$. That is, we can make $\| f - \Psi \|_{W^{p,k}(\Omega \cup \partial\Omega, Cl_n)}$ as small as we please.

Therefore, for $\epsilon > 0$, and A an index set, from the above argument, we can make a component-wise $\mathbb{R}-$valued smooth approximation

$$\| f_A - \psi_A \|_{W^{p,k}} < \frac{\epsilon^p}{2^{np}}$$

on $\Omega \cup \partial\Omega$. The factor 2^{-n} in the last inequality is related to the cardinality of a basis of the Clifford algebra Cl_n.

Then considering the functions Ψ and f, with corresponding component functions with the above corresponding sectional smooth approximations, we have :

$$\| f - \Psi \|_{W^{p,k}(\Omega \cup \partial\Omega, Cl_n)} = \| \sum_A e_A (f_A - \psi_A) \|_{W^{p,k}(\Omega \cup \partial\Omega, Cl_n)}$$

$$= \left(\sum_A (\| f_A - \psi_A \|^p_{W^{p,k}(\Omega \cup \partial\Omega,\mathbb{R})}) \right)^{p^{-1}} < \epsilon$$

□

3. **Applications**

In this section, we see the application of the two methods we discussed above : approximation of a $\gamma-$regular function by minimal family of functions and approximation of such a function by smooth functions.

The application of the complete and minimal function systems that we constructed in approximating null solutions of first order partial differential equations of the in-homogeneous Dirac operator is presented in the following proposition.

Proposition 4. [6]*Let* Ω *and* g *be as in proposition* 2 *. Then for a given* $\varepsilon > 0$ *and for a given left* $\gamma-$*regular function* f *given as a solution of the BVP*(1.5) *in proposition* 2 *, there exist Clifford numbers* $\beta_j (j = 1, ...n_0)$ *such that*

$$\| f - \sum_{j=1}^{n_0} \Psi_j^{\Gamma} \beta_j \|_{W^{2,k}_{\Gamma, Cl_n}} < \varepsilon$$

on Ω*.*

Proof. Since the system $\{\Psi_m^{\Gamma}(x) := \frac{\overline{(x-x_m)}}{\omega_n \|x-x_m\|^n} e^{-\Gamma(x-x_m)}\}_m$ is Cl_n-complete in the space of left γ-regular functions which are in $W^{2,k}_{\Gamma}(\Omega, Cl_n)$, where $\{x_m\}_m$ is a dense subset of some outer hypersurface Σ_{out} of the domain Ω such that $dist(\Sigma_{out}, \partial\Omega) \geq \delta > 0$, the solution f of the BVP(1.5) in proposition 2, can be approximated with finitely many elements of $\{\Psi_m^{\Gamma}\}_m$. That means, $\exists \beta_j \in Cl_n$ $(j = 1, ..., n_0)$ such that the above approximation inequality holds. The Clifford numbers $\beta_j (j = 1, ...n_0)$ are determined by solving a system of equations obtained from the boundary conditions

$$tr_{\Sigma} \sum_{j=1}^{n_0} \Psi_j^{\Gamma} \beta_j(y_i) = g(y_i)$$

for each $i = 1, ..., n_0$, where $\{y_i : i = 1, ..., n_0\}$ is a set of unisolvent points selected on Σ as in proposition 9. □

Then a best approximation of the above solution can be obtained from the minimal functions.

Corollary 1. [6]*Using the* Cl_n- *minimal functions* $\{\phi_k\}_k$*, the solution* f *given by equation* (1.6) *of the* ***BVP*** (1.5) *is approximated in the best*

way in $B_{(n0)} = \underset{Cl_n}{span}\left(\{\phi_j\}_{j=1}^{n_0}\right)$ *as*

$$\| f - \sum_{j=1}^{n_0} \phi_j \lambda_j \|_{W_\Gamma^{2,k}} < \varepsilon$$

with λ_j $(j = 1, ..., n_0)$ *determined as in proposition* 11.

The next proposition gives the smooth approximation of a null solution of the in homogeneous Dirac operator which is in a certain Sobolev space.

Proposition 5. *Let* Ω *and* g *be as in proposition* 2 *. Then for a given* $\varepsilon > 0$ *and for a given left* $\gamma-$*regular function* f *given in* (1.6) *as a solution of the BVP* (1.5) *in proposition* 2*, there exists a* $C^\infty-$ *function* $\Psi = \sum_A e_A \psi_A$ *over* $\Omega \cup \partial\Omega$ *such that*

$$\| f - \Psi \|_{|W_\Gamma^{2,k}(\Omega \cup \partial\Omega, Cl_n)} < \varepsilon.$$

Proof. The analytic solution of the BVP(1.5) is given by a boundary integral (1.6) and this boundary integral which is also written as $f = F_{\partial\Omega}(g) = F_{\partial\Omega}(tr_{\partial\Omega} f)$ puts the solution in to the Sobolev space $W^{2,k}(\Omega, Cl_n)$.

This is because the trace operator as a sharpening operator (that reduces smoothness in this case by a $\frac{1}{2}$) has the property :

$$tr_{\partial\Omega} : W^{2,k}(\Omega, Cl_n) \to W^{2,k-\frac{1}{2}}(\partial\Omega, Cl_n)$$

and the $\partial-$ integral as a left inverse of the $tr_{\partial\Omega}-$operator as a mapping where the argument is a $\gamma-$regular function, is a smoothening operator with the property :

$$F_{\partial\Omega} = \left(\Psi^\Gamma \underbrace{*}_{\text{convolution}} \nu(\cdot) \right)_{|\partial\Omega} : W^{2,s}(\partial\Omega, Cl_n) \to W^{2,s+\frac{1}{2}}(\Omega, Cl_n)$$

where, ν is the unit normal vector function defined on the boundary of Ω.

But in general, the two operators, $\partial-$integral and $tr_{\partial\Omega}-$ are inverses of each other in terms of preserving regularity, not as function transformations.

Therefore, the solution function f which is Cl_n-valued can be written as
$f = \sum_A e_A f_A$, with

$$f_A := \left(\int_{\partial\Omega} \Psi_\Gamma(x-y)\nu(y)g(y)d\Sigma_y \right)_A$$

the $A-$component of f.

But then as above, there exists a corresponding smooth Clifford valued function ψ_A so that for $\epsilon > 0$, we have

$$\| f_A - \psi_A \|_{W^{2,k}(\Omega\cup\partial\Omega,\mathbb{R})} \leq \frac{\epsilon^2}{2^{2n}}$$

Therefore, by taking Ψ as the Clifford sum of the component functions ψ_A, we have the following inequality:

$$\begin{aligned} \| \quad f - \Psi \|_{W^{2,k}(\Omega\cup\partial\Omega,Cl_n)} &= \| \sum_A e_A \left(f_A - \psi_A \right) \|_{W^{2,k}(\Omega\cup\partial\Omega,Cl_n)} \\ \leq \quad \sum_A \frac{\epsilon}{2^n} &< \epsilon \end{aligned}$$

□

Interestingly enough, the smooth approximation works to BVPs which have non-vanishing Dirac derivatives over the domain, unlike the minimal family approximation which we have only for $\gamma-$regular functions with a non vanishing trace.

We therefore give this result in the following proposition.

Proposition 6. *Let Ω be a bounded domain in $\mathbb{R}^n$ whose boundary is a C^2- hyper surface and let $g \in W^{2,k-1}(\Omega, Cl_n)$, $h \in W^{2,k-\frac{1}{2}}(\partial\Omega, Cl_n)$, then the:*

$$\mathbf{NHBVP}: \left\{ \begin{array}{c} D_\gamma f = g, \text{ on } \Omega \\ trf = h, \text{ on } \partial\Omega \end{array} \right.$$

has a solution f which is in the Sobolev space $W^{2,k}(\Omega, Cl_n)$ given by

$$f(x) = \int_{\partial\Omega} \Psi^\Gamma(x-y)\, v(y)\, h(y)\, d\partial\Omega_x + \int_\Omega \Psi^\Gamma(x-y)\, g(x)\, d\Omega_x$$

and therefore there exists a sequence $\{\varphi_m\}_{m=1}^{\infty} \subset C^{\infty}(\Omega \cup \partial\Omega, Cl_n)$ *such that for* $\epsilon > 0, \exists n_0 \in \mathbb{N} \ni$

$$\|\varphi_k - \left(\int_{\partial\Omega} \Psi^{\Gamma}(x-y)\, v(y)\, h(y)\, d\partial\Omega_x + \int_{\Omega} \Psi^{\Gamma}(x-y)\, g(x)\, d\Omega_x \right) \|_{W^{2,k}(\Omega\cup\partial\Omega, Cl_n)} < \epsilon$$

for all $k \geq n_0$.

Proof. First, one can see that the $\Omega-$ integral has the mapping property:

$$\left(\Psi^{\Gamma} \underbrace{*}_{\text{convolution}} (\cdot) \right)_{|\Omega} : W^{2,k}(\Omega, Cl_n) \to W^{2,k+1}(\Omega, Cl_n)$$

which is a smoothness augmentation by a one unlike the $\partial-$ integral which increases by a half.

Next, let

$$f_A := \left(\int_{\partial\Omega} \Psi^{\Gamma}(x-y)\, v(y)\, h(y)\, d\partial\Omega_x + \int_{\Omega} \Psi^{\Gamma}(x-y)\, g(x)\, d\Omega_x \right)_A$$

the $A-$ Clifford section of f.

Then $f_A : \Omega \to \mathbb{R}$ is in the Sobolev section $W^{2,k}(\Omega, \mathbb{R})$ and therefore, $\exists$ a sequence $\left\{\varphi_{A,j}\right\}_{j=1}^{\infty} \subseteq C^{\infty}(\Omega \cup \partial\Omega, \mathbb{R})$ such that for $\epsilon > 0$, $\exists n_A \in \mathbb{N}$ such that for $m_A \geq n_A$, where $m_A \in \mathbb{N}$, we have

$$\|\varphi_{A,m_A} - f_A\|_{W^{2,k}(\Omega\cup\partial\Omega,\mathbb{R})} < \frac{\epsilon^2}{2^{2n}}$$

Then for $n_0 := \max\{m_A : A \text{ is an index set}\}$ and for $k \geq n_0$, taking the Clifford valued function given by $\varphi_k := \sum_A e_A \varphi_{A,k}$ which is $C^{\infty}-$over $\Omega \cup \partial\Omega$, we have

$$\begin{aligned} \|f - \varphi_k\|_{W^{2,k}(\Omega\cup\partial\Omega,Cl_n)} &= \|\sum_A e_A \left(f_A - \varphi_{A,k}\right)\|_{W^{2,k}(\Omega\cup\partial\Omega,Cl_n)} \\ &\leq \sum_A \frac{\epsilon}{2^n} < \epsilon \end{aligned}$$

that proves the proposition. □

The next results focus on how far away are solutions of NHBVPs stated in proposition 6, from space of monogenic functions or $\gamma-$regular functions defined over the domain Ω, if the input functions g and h are $C^{\infty}-$ over the respective domains of definition. We first put Alexander's inequality for our purpose.

Proposition 7. *(Alexander)Let f be a Clifford valued C^{∞}- function defined over a compact domain Ω in $\mathbb{R}^{n+1}$.*

Then

$$\underset{C(\Omega,Cl_n)}{dist}\left(f, M\left(\Omega, Cl_n\right)\right) \leq \beta\left(\mu\left(\Omega\right)^{\left(\frac{1}{n+1}\right)}\right)\|Df\|_{\infty}$$

where, μ is the volume measure in $\mathbb{R}^{n+1}$ and $\|\cdot\|_{\infty}$ is the supremum norm and $M\left(\Omega, Cl_n\right)$ is the set of Clifford valued functions defined over Ω which are annihilated by the Dirac differential operator D.

From the above result of Alexander, we get the following important inequality on solutions of NHBVPs.

Proposition 8. *Let Ω be a compact domain in $\mathbb{R}^{n+1}$ and g be a $C^{\infty}-$function over Ω and h also be $C^{\infty}-$over $\partial\Omega$. Then the solution to the NHBVP:*

$$\begin{cases} D_{\gamma}f = g, \text{ on } \Omega \\ trf = h, \text{ on } \partial\Omega \end{cases}$$

satisfies the inequality:

$$\underset{C(\Omega,Cl_n)}{dist}\left(f, M_{\gamma}\left(\Omega, Cl_n\right)\right) \leq \beta\left(\mu\left(\Omega\right)^{\left(\frac{1}{n+1}\right)}\right)\|g\|_{\infty}$$

where, $M_{\gamma}\left(\Omega, Cl_n\right)$ is the set of Clifford valued functions defined over Ω which are annihilated by the Dirac like Differential operator D_{γ}.

Proof. From Borel-Pompeiu relation, the solution to the NHBVP given above is given by the following integral equation:

$$f = \int\limits_{\partial\Omega} \Psi^{\Gamma}\left(x-y\right)\nu trf d\partial\Omega + \int\limits_{\Omega} \Psi^{\Gamma}\left(x-y\right) D_{\gamma}f d\Omega$$

Using the input functions given on the domain and on the boundary, we have the solution function to be :

$$f = \int\limits_{\partial\Omega} \Psi^{\Gamma}\left(x-y\right)\nu h d\partial\Omega + \int\limits_{\Omega} \Psi^{\Gamma}\left(x-y\right) g d\Omega$$

Then by the inequality of Alexander, we have :

$$\underset{C(\Omega,Cl_n)}{dist}\left(\int_{\partial\Omega}\Psi^{\Gamma}(x-y)\,\nu h d\partial\Omega+\int_{\Omega}\Psi^{\Gamma}(x-y)\,g d\Omega, M(\Omega,Cl_n)\right)\leq\beta\left(\mu(\Omega)^{\left(\frac{1}{n+1}\right)}\right)\|g\|_{\infty}$$

□

Remark 1. *From the above inequality, one can see that if the domain is of measure zero, then the solution is always approximated by monogenic functions, as the indicated distance of the solution function from the family of monogenic functions defined over Ω is zero for such a set. Besides, if the input function g has a zero supremum norm then we have also similar results.*

But in a softer note, we see a very important relation between the supremum norm of the input function g and how far is the solution function away from monogenic functions. The thicker the supremum norm of the input function, the farther away is the solution of the NHBVP from being a monogenic function.

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Virginia Union University, Department of Mathematics & Computer Science
E-mail address: dalakew@vuu.edu
URL: http://www.vuu.edu