■ : A Note on Differentiability of a Function:

Given a function f , a domain D and a number $a \in D$.

We say that : f is differentiable at a or : the derivative of f at a exists, if the limit:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. We denote the above limit by $f'(a)$: which is read as " f prime of a ".

$$\therefore f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

♣ : Geometrically, differentiability of a function at a is precisely saying that there is only one tangent line to the graph of the function at the point $(a, f(a))$ which is indicated in the graph below:

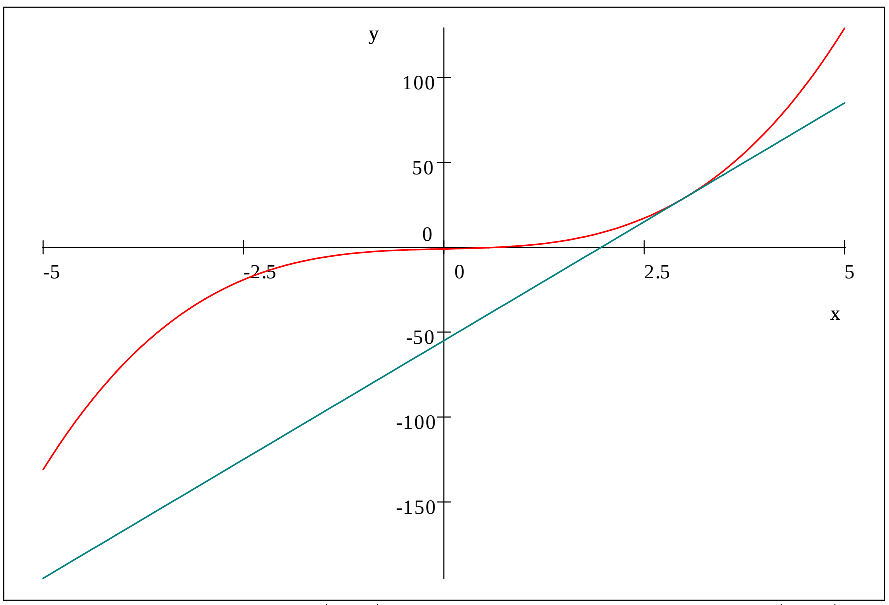

Fig. 1 The graph of f (red) and the unique tangent line $l_{\tan}$(blue)

♣ :Alternative description of the above limit:

We make the substitution : $x - a = h$. Then three things follow :

i. $x = a + h$

ii. $x \to a$ is the same as $h \to 0$

iii. $\therefore f'(a) := \lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$

When we find the derivative of the function at any number $x \in D$, we do :

$$f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

This limit is called the derivative of f at x which is a function of x.

$\lessdot$:**Non-Differentiability case**:

If the limit : $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim\limits_{h\to 0} \frac{f(a+h)-f(a)}{h}$ does not exist, then we say the function f is not differentiable at a and write:
$f'(a)\nexists$ or $f'(a)$ does not exist.

The symbol $\nexists$ is to say does not exist.

We therefore pose a question as to what causes a function not to be differentiable at a given number a ?

Among the many reasons, I put few of them :

1. If the function is not continuous at the number a.
2. If the function has a *cusp* point which looks like : $\curlywedge$,$\curlyvee$ or a *corner* point : $\vee, \wedge$ on the graph at $(a, f(a))$
3. If the function has a vertical tangent line to the graph at : $(a, f(a))$.

Example 1 *(Cusp point) The function given by :*

$$f(x) = \begin{cases} (x-2)^2 & \text{if} \quad 1 \le x \le 7 \\ x^2 & \text{if} \quad -5 \le x \le 1 \end{cases}$$

is not differentiable at $a = 1$ where the graph has a cusp $\curlywedge$ point at $(1, 1)$

Solution 2 *Clearly the function is continuous at $a = 1$. So we have to check for the other conditions.*
Existence of a cusp or corner point or vertical tangent line at $(1, f(1))$.

Indeed, the one sided derivatives tell us that:

The right hand derivative of f at 1 denoted by $f'(1^+)$ is

$$f'(1^+) := \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{(x-2)^2 - 1}{x - 1}$$

$$= \lim_{x \to 1^+} \frac{(x-1)(x-3)}{x-1} = \lim_{x \to 1^+} (x-3) = -2$$

And the left hand derivative of f at 1 denoted by $f'(1^-)$ is :

$$f'(1^-) := \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x+1) = 2$$

That is : $f'(1^+) = -2 \neq 2 = f'(1^-)$

The two sided derivatives of f at $a = 1$ are different.

$$\therefore f'(1) \nexists$$

If you sketch the graph of the function you see that the point $(1, f(1)) = (1, 1)$ is indeed a cusp point : a point of the type: 人.

Look the graph below.

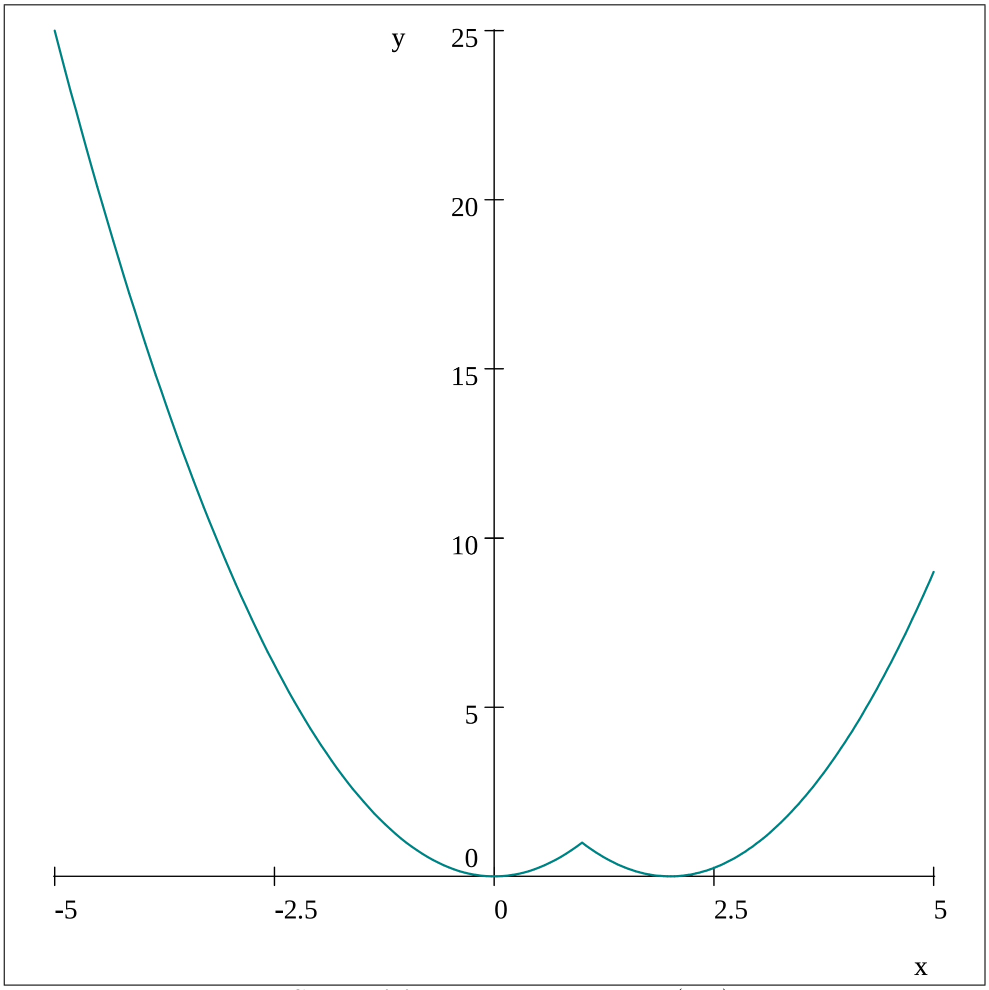

Fig. 2 Graph of f *with a* cusp *point at* $(1,1)$

Example 3 *(Corner point) The function :*

$$f(x) = \begin{cases} x & if \quad x \geq 0 \\ -x & if \quad x < 0 \end{cases}$$

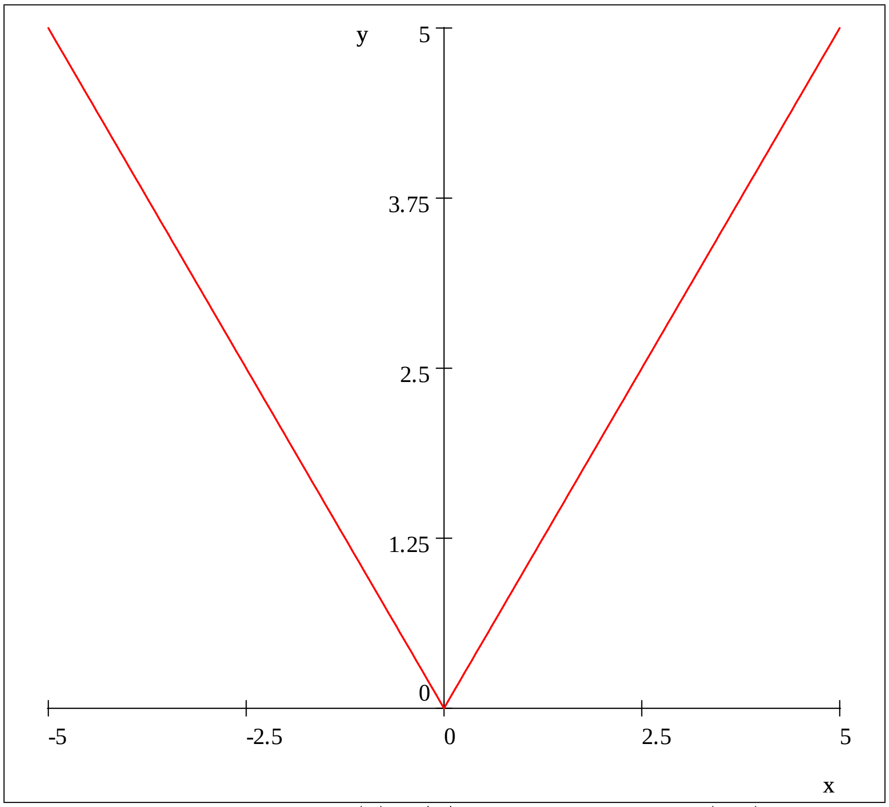

Fig. 3 *Graph of* $f(x) = |x|$ *with* corner *point at* $(0,0)$

is the usual absolute value function where the graph has a corner point $\vee$ *at* $(0,0)$

We have seen in class that the function is not differentiable at $a = 0$ *as the right hand derivative and left hand derivative of* f *at* 0 *are unequal.*

Example 4 *(Vertical Tangent Line) The cube root function:*

$$f(x) = \sqrt[3]{x}$$

is continuous at $a = 0$ *but not differentiable there.*

Indeed ,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{\sqrt[3]{x} - 0}{x} = \lim_{x \to 0} \frac{\sqrt[3]{x}}{x} = \lim_{x \to 0} \frac{1}{\sqrt[3]{x^2}} = \infty$$

Which implies that the graph has a vertical tangent line at that point. Look the graph below.

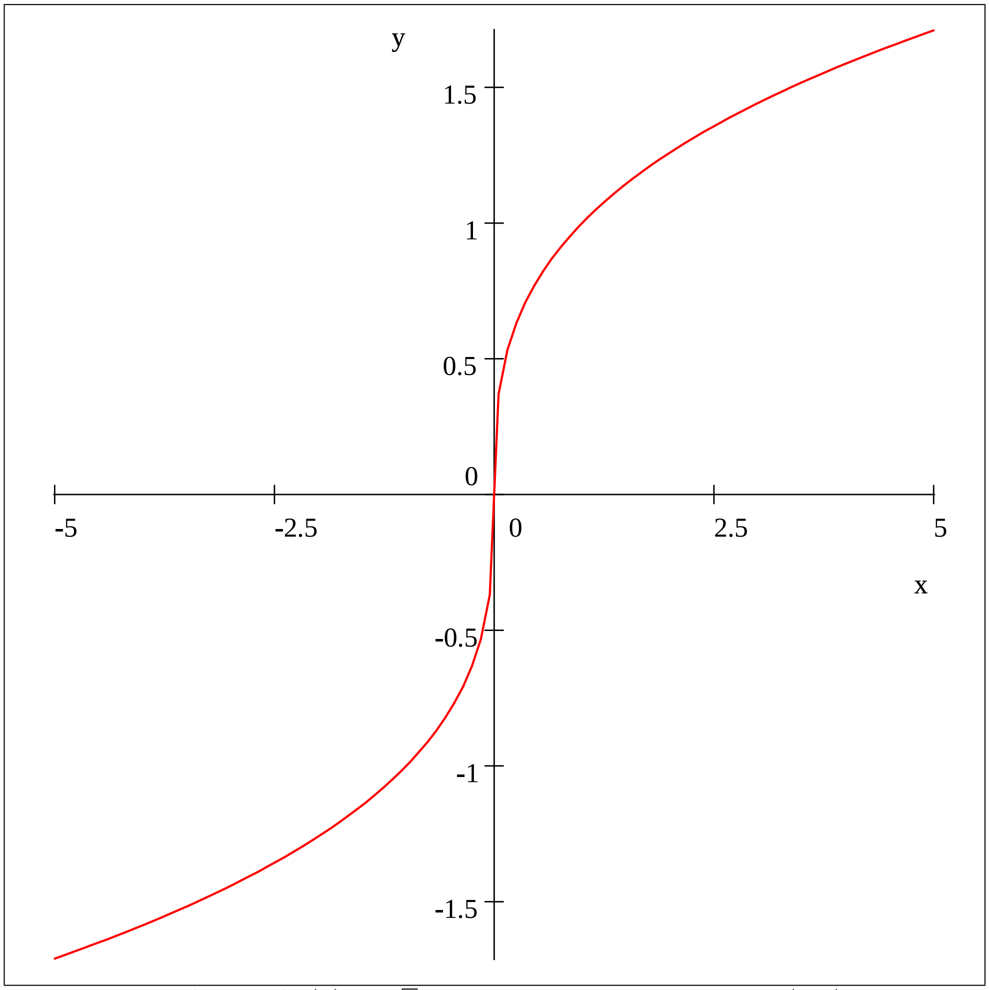

Fig. 4 Graph of $f(x) = \sqrt[3]{x}$ with a vertical tangent *line at* $(0,0)$

Example 5 *(At a discontinuity point)* (★) *:A discontinuous function f at a number a can not be differentiable at that number.*

Reason*: We have proved a result in class that :*

(✠) *:A differentiable function at a number is continuous there .*

Therefore the statement given in (★) *is in fact a contrapositive of the result* (✠)

Exercise 6 *Show that the function given is not differentiable at the indicated number and give a geometrical explanation:*

1. $f(x) = |x - 5|, a = 5$
2. $g(x) = \sqrt[3]{x + 2}, a = -2$
3. $k(x) = \begin{cases} 2 - (x - 2)^2 & \text{if} \quad x \geq 1 \\ x^2 & \text{if} \quad x \leq 1 \end{cases}, a = 1$

   ***Hint**: Show the one sided derivatives of the functions at the given numbers are unequal and there by conclude the non-diff$\frac{bility}{}$ there.*