This is my fifth posting in the thread :

Communications in Mathematics Teaching (CMT) as a series :

This is a differential operator of even orders with infinite terms defined below :

$$D^{\infty,even} := \sum_{k=0}^{\infty} \frac{D^{(2k)}}{(2k)!}$$

where $D := \frac{d}{dx}$

Then as in my previous communications, we question the following:

$(\forall \psi \epsilon C^{\infty}(I,\mathbb{R})) \Lambda (\forall x \epsilon I)$, what will be

$$D^{\infty,even}(\psi(x)) = \sum_{k=0}^{\infty} \frac{D^{(2k)}\psi(x)}{(2k)!}?$$

As my favorite example, let us consider the following:

Example 1: The natural exponential function: $\psi(x) = e^x$

Claim: $D^{\infty,even}(\psi(x)) = \psi(x+1) + \psi(x-1)$.

Indeed,

$$\begin{aligned} D^{\infty,even}(\psi(x)) &= \sum_{k=0}^{\infty} \tfrac{D^{(2k)}\psi(x)}{(2k)!} \\ &= \sum_{k=0}^{\infty} \tfrac{D^{(2k)}(e^x)}{(2k)!} \\ &= e^x \left(\sum_{k=0}^{\infty} \tfrac{1}{(2k)!} \right) \\ &= e^x \left(\sum_{k=0}^{\infty} \tfrac{1}{(k)!} + \sum_{k=0}^{\infty} \tfrac{(-1)^k}{(k)!} \right) \\ &= e^{x+1} + e^{x-1} \\ &= \psi(x+1) + \psi(x-1) \end{aligned}$$

Example 2. Let $\xi(x) = x^3$.Then

$$
\begin{aligned}
D^{\infty,even}\xi(x) &= \sum_{k=0}^{\infty} \frac{D^{(2k)}\xi(x)}{(2k)!} \\
&= \sum_{k=0}^{\infty} \frac{D^{(2k)}\left(x^3\right)}{(2k)!} \\
&= 3x^3 + 6x
\end{aligned}
$$

But the expression we have at the end is precisely: $\xi\left(x+1\right)+\xi\left(x-1\right)$

$$
\therefore \qquad D^{\infty,even}\xi(x) = \xi(x+1) + \xi\left(x-1\right)
$$

In a similar way:

Lemma: For the monomial : $\xi(x) = x^n$, $D^{\infty,even}\xi(x) = \xi\left(x+1\right) + \xi\left(x-1\right)$
Proof of Lemma: Here one can show that :
$e^D\left(\xi\left(x\right)\right) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) x^{n-k} = \left(x+1\right)^n = \xi(x+1)$
and the result follows.

Proposition: $\forall\ p(x)\epsilon\wp\left(x\right)$, $D^{\infty,even}p(x) = p(x+1) + p(x-1)$, where $\wp\left(x\right)$ is the set of all polynomial functions in x

Conjecture : $\forall\psi\epsilon C^{\infty}\left(I,\mathbb{R}\right)$, $D^{\infty,even}\psi\left(x\right) = \psi\left(x+1\right) + \psi\left(x-1\right)$

Also for each positive integer k, consider the differential operator of the type:

$$
D_k^{\infty,even} := \sum_{j=0}^{\infty} \frac{k^{(2j)}D^{(2j)}}{(2j)!}
$$

where D is the differential operator as above. Then we can write the following interesting statement:

Conjecture: $\left(\forall k\epsilon\mathbb{N}\right)\left(\forall\psi\epsilon C^{\infty}\left(I,\mathbb{R}\right)\right), D_k^{\infty,even}\psi\left(x\right) = \psi\left(x+k\right) + \psi\left(x-k\right)$

Note: One can show the validity of following results:

(i) $D^{\infty,even}\left(\sin(x)\right) = 2\sin x\cos 1$
(ii) $D^{\infty,even}\left(\cos(x)\right) = 2\cos x\cos 1$

Remark: The differential operators I considered in four of my communications can have some sort of mapping properties. One can look at these things.

Next time I will bring some properties of the operators on trigonometric functions. Who knows they might have some interesting relations.This is my fifth posting in the thread :

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