On Some Discrete Differential Equations

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In differential equations class, we teach the Laplace transform as one of the tools available to find solutions of linear differential equations of constant coefficients. What we do in this note is to use a non-continous Laplace transform, that generates sequential solutions which are polynomials in $\mathbb{N}$ or quotients of such polynomials. We use the following definitions. For a $\mathbb{R}$-valued sequence $f : \mathbb{N} \to \mathbb{R}$,

Definition 1 *The discrete Laplace transform of $f(n)$ is defined as*

$$\ell_d \{f(n)\}(s) := \sum_{n=0}^{\infty} e^{-sn} f(n), \text{ where } s > 0.$$

Definition 2 *The first order difference equation of a sequence $f(n)$ is defined as*

$$\Delta f(n) := f(n+1) - f(n)$$

Proposition 3 *The first order discrete IVP: $\triangle f(n) = n$, $f(1) = 1$, has solution given by*

$$f(n) = 1 + \frac{n^2 - n}{2}.$$

Proof. Taking the transform of both sides of the equation : $\ell_d \{\triangle f(n)\}(s) = \ell_d \{n\}$, we get

$$(e^s - 1)\, \ell_d \{f(n)\}(s) - f(1) = \frac{e^s}{(e^s - 1)^2}.$$

Substituting the value $f(1) = 1$, and simplifying the expression we get,

$$\ell_d \{f(n)\}(s) = \frac{1}{e^s - 1} + \frac{e^s}{(e^s - 1)^3}.$$

Then taking the inverse transform we have the solution :

$$f(n) = \ell_d^{-1} \left\{ \frac{1}{e^s - 1} + \frac{e^s}{(e^s - 1)^3} \right\}$$

$$= 1 + (n * 1) = 1 + \frac{n^2 - n}{2}.$$

Here "$*$ " is the convolution operator. ∎

Proposition 4 *The second order IVP:* $\triangle^2 f(n) = n,\ f(1) = 1,\ \triangle f(1) = 2,$ *has solution given by :*

$$f(n) = 2n - 1 + \frac{n(n-1)(n-2)}{6}.$$

Proof. First,

$$\triangle^2 f(n) = \triangle(\triangle f(n)) = f(n+2) - 2f(n+1) + f(n)$$

and using the initial conditions we get:

$$\ell_d\left\{\triangle^2 f(n)\right\}(s) = \left(e^{2s} - 2e^s + 1\right)\ell_d\{f(n)\}(s) - e^s - 1.$$

$$\Rightarrow (e^s - 1)^2\,\ell_d\{f(n)\}(s) - e^s - 1 = \frac{e^s}{(e^s-1)^2}$$

$$\Rightarrow \ell_d\{f(n)\}(s) = \frac{1}{(e^s-1)^2} + \frac{e^s}{(e^s-1)^2} + \frac{e^s}{(e^s-1)^4}.$$

Then taking the inverse transform and using convolutions, we get the solution as :

$$f(n) = (1 * 1) + n + \frac{n(n-1)(n-2)}{6}$$

$$= 2n - 1 + \frac{n(n-1)(n-2)}{6}.$$

∎

Proposition 5 $\ell_d\left\{\frac{1}{n}\right\}(s) = s - \ln(e^s - 1)$ *for* $s > 0$.

Proof.

$$\ell_d\{1\}(s) = \frac{1}{e^s - 1} = \sum_{n=1}^{\infty} e^{-sn} \quad \text{for } s > 0.$$

Integrating both sides, we get

$$\ln(e^s - 1) - s = \sum_{n=1}^{\infty}\left(-\frac{e^{sn}}{n}\right) = \ell_d\left\{\frac{-1}{n}\right\}(s)$$

$$\Rightarrow \ell_d\left\{\frac{1}{n}\right\}(s) = s - \ln(e^s - 1).$$

∎

We now present discrete IVPs whose solutions are rational sequences in n.

Proposition 6 *The discrete IVP :* $n\triangle f(n) = 1,\ f(2) = 2$*, for* $n \geq 2$ *has solution given by:*

$$f(n) = 1 + \sum_{k=1}^{n-1} \frac{1}{k}.$$

Proof. Taking the transform of both sides of the equation, we have:

$$\ell_d \{n\triangle f(n)\}(s) = \frac{1}{e^s - 1}.$$

But

$$\ell_d \{n\triangle f(n)\}(s) = \ell_d \{nf(n+1) - nf(n)\}(s)$$

$$= e^s \ell_d \{nf(n)\}(s) - e^s \ell_d \{f(n)\}(s) - \ell_d \{nf(n)\}(s)$$

$$= (e^s - 1)\ell_d \{nf(n)\}(s) - e^s \ell_d \{f(n)\}(s).$$

Thus,

$$(e^s - 1)\ell_d \{nf(n)\} - e^s \ell_d \{f(n)\} = \frac{1}{e^s - 1}.$$

Again,

$$\ell_d \{nf(n)\}(s) = -\frac{d}{ds}\ell_d \{f(n)\}(s).$$

Therefore,

$$-(e^s - 1)\frac{d}{ds}\ell_d \{f(n)\}(s) - e^s \ell_d \{f(n)\}(s) = \frac{1}{e^s - 1},$$

which is an ordinary non-homogenious linear differential equation of first order in s and writing it in standard form we have :

$$\frac{d}{ds}\ell_d \{f(n)\}(s) + \frac{e^s}{e^s - 1}\ell_d \{f(n)\}(s) = -\frac{1}{(e^s - 1)^2}$$

whose solution for $\ell_d \{f(n)\}(s)$ is given by $\frac{1}{e^s-1} - \frac{\ln(e^s-1)-s}{e^s-1}$.
Then taking the inverse transform , we have :

$$f(n) = \ell_d^{-1}\left\{\frac{1}{e^s - 1} + \frac{s - \ln(e^s - 1)}{e^s - 1}\right\}$$

$$= 1 + \ell_d^{-1}\{s - \ln(e^s - 1)\} * \ell_d^{-1}\left\{\frac{1}{e^s - 1}\right\}$$

$$= 1 + \left(\frac{1}{n} * 1\right) = 1 + \sum_{k=1}^{n-1} \frac{1}{k} \quad \text{for } n > 1.$$

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Proposition 7 *For $n \geq 2$, the IVP : $\triangle f(n) = \frac{1}{n^2}$, $f(2) = 2$ has solution given by*

$$f(n) = 1 + \sum_{k=1}^{n-1} \frac{1}{k^2}.$$

Proof. Re-writing the difference equation as : $n\triangle f(n) = \frac{1}{n}$, taking the transform of both sides and using corollary 3.3 we get

$$\frac{d}{ds}\ell_d \{f(n)\}(s) + \frac{e^s}{e^s - 1}\ell_d \{f(n)\}(s) = -\frac{s - \ln(e^s - 1)}{e^s - 1}.$$

Again solving for $\ell_d \{f(n)\}(s)$, we have

$$\ell_d \{f(n)\}(s) = \frac{1}{e^s - 1} - \frac{1}{e^s - 1}\int (s - \ln(e^s - 1))\, ds.$$

$$\Rightarrow f(n) = 1 - \ell_d^{-1}\left\{\frac{1}{e^s - 1}\right\} * \ell_d^{-1}\left\{\int (s - \ln(e^s - 1))\, ds\right\}$$

$$= 1 - \left(1 * \left(-\frac{1}{n^2}\right)\right) = 1 + \sum_{k=1}^{n-1} \frac{1}{k^2}.$$

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