Discrete Differential Equations and $\sum$-Transform

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Abstract. In this short paper we define a dicrete Lapace transform, list some properties of it and solve some first and second order discrete differential equations or simply called difference equations whose solutions are polynomials of integers or their quotients.

1. $\sum$-Transform: Definition and Examples.

Laplace transform is one of the fine tools available to solve linear differential equations with constant coefficients. In this short note, we introduce a $\sum$-transform(or a discrete transform), develop some of its properties and see its applications in solving discrete differential equations or simply difference equations of special types: difference equations whose solutions are polynomials of positive integers or their quotients. In the sequel, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{R}$ denotes the set of all real numbers and IVP stands for Initial Value Problem.

Definition 1. *Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence and let $s > 0$. We define the discrete Laplace transform of f by $\ell_d \{f(n)\}(s) := \sum_{n=1}^{\infty} f(n) e^{-sn}$, provided the series converges.*

Example 1. $\ell_d \{1\}(s) = \frac{1}{e^s - 1}$ *for $s > 0$.*

Example 2. $\ell_d \{n\}(s) = \frac{e^s}{(e^s-1)^2}$, $s > 0$. *Indeed from the geometric series $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ for $\mid x \mid < 1$, we have*

$$\begin{aligned} \sum_{n=1}^{\infty} nx^n &= x\left[\frac{d}{dx}\sum_{n=1}^{\infty} x^n\right] \\ &= \frac{x}{(1-x)^2}. \end{aligned}$$

2000 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Discrete Laplace Transform, Descrete Differential Equations .

Therefore,

$$\begin{aligned}\ell_d\left\{n\right\}(s) &= \sum_{n=1}^{\infty} ne^{-sn} \\ &= \frac{e^s}{\left(e^s-1\right)^2}.\end{aligned}$$

EXAMPLE 3. $\ell_d\left\{n^2\right\}(s)=\frac{e^{2s}+e^s}{\left(e^s-1\right)^3}$. *Here again, for* $-1<x<1:$

$$\begin{aligned}\sum_{n-1}^{\infty} n^2x^n &= x\left(\frac{d}{dx}\left(x\frac{d}{dx}\sum_{n=1}^{\infty}x^n\right)\right) \\ &= \frac{x^2+x}{\left(1-x\right)^3}.\end{aligned}$$

Hence

$$\begin{aligned}\ell_d\left\{n^2\right\}(s) &= \sum_{n=1}^{\infty} n^2e^{-sn} \\ &= \frac{e^{2s}+e^s}{\left(e^s-1\right)^3}.\end{aligned}$$

By performing two operations one after the other: differentiating and multiplying by x on $\sum_{n=1}^{\infty}x^n=\frac{x}{1-x}$ for $-1<x<1$, we get the following results:

▲
$$\ell_d\left\{n^3\right\}(s)=\frac{e^{3s}+4e^{2s}+e^s}{\left(e^s-1\right)^4}$$

▲
$$\ell_d\left\{n^4\right\}(s)=\frac{e^{4s}+11e^{3s}+11e^{2s}+e^s}{\left(e^s-1\right)^5}$$

▲
$$\ell_d\left\{n^5\right\}(s)=\frac{e^{5s}+26e^{4s}+66e^{3s}+26e^{2s}+e^s}{\left(e^s-1\right)^6}$$

▲
$$\ell_d\left\{n^6\right\}(s)=\frac{e^{6s}+57e^{5s}+302e^{4s}+302e^{3s}+57e^{2s}+e^s}{\left(e^s-1\right)^7}$$

PROBLEM 1. *What is* $\ell_d\left\{n^k\right\}(s)$ *for any* $k\in\mathbb{N}$ *?*

2. Existence and some properties of the ℓ_d-transform.

Let $f(n):\mathbb{N}\to\mathbb{R}$ be a sequence such that $\mid f(n)\mid\leq\alpha e^{s_0n}$ for $\alpha>0$, $s_0>0$. Then $\sum_{n=1}^{\infty}f(n)\,e^{-sn}$ is absolutely convergent and hence is convergent. Therefore, for such a sequence, the discrete Laplace transform $\ell_d\left\{f(n)\right\}(s)$ exists finitely for $s>s_0$, since

$$\mid\sum_{n=1}^{\infty}f(n)\,e^{-sn}\mid\leq\sum_{n=1}^{\infty}\alpha e^{(s_0-s)n}=\frac{\alpha}{e^{s-s_0}-1}<+\infty$$

for $s > s_0$. From this, we conclude that sequences which are polynomials in n have discrete Laplace transforms.

LEMMA 1. *ℓ_d and its inverse ℓ_d^{-1} are both linear.*

PROPOSITION 1. *(Transform of translate of a sequence). For $k \in \mathbb{N}$,*

$$\ell_d \{f(n+k)\}(s) = e^{ks}\ell_d \{f(n)\}(s) - \sum_{i=1}^{k} f(i) e^{(k-i)s}.$$

PROOF. Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence. Then

$$\begin{aligned} \ell_d \{f(n+k)\}(s) &= \sum_{n=1}^{\infty} f(n+k) e^{-sn} \\ &= \sum_{m=k+1}^{\infty} f(m) e^{-s(m-k)} \end{aligned}$$

$$= e^{sk} \sum_{m=k+1}^{\infty} f(m) e^{-sm} = e^{ks} \sum_{m=1}^{\infty} f(m) e^{-sm} - \sum_{i=1}^{k} f(i) e^{(k-i)s}$$

$$= e^{ks}\ell_d \{f(n)\}(s) - \sum_{i=1}^{k} f(i) e^{(k-i)s}.$$

□

COROLLARY 1. *$\ell_d \{f(n+1)\}(s) = e^{s}\ell_d \{f(n)\}(s) - f(1)$.*

DEFINITION 2. *Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence. The discrete derivative of of f denoted*

$$\triangle f(n) := f(n+1) - f(n).$$

PROPOSITION 2. *(Transform of of a discrete derivative of a sequence).*

$$\ell_d \{\triangle f(n)\}(s) = (e^s - 1) \ell_d \{f(n)\} - f(1).$$

PROOF.

$$\ell_d \{\triangle f(n)\}(s) = \sum_{n=1}^{\infty} \triangle f(n) e^{-sn} = \sum_{n=1}^{\infty} (f(n+1) - f(n)) e^{-sn}$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} f(n+1) e^{-sn} - \sum_{n=1}^{\infty} f(n) e^{-sn} \\ &= \ell_d \{f(n+1)\} - \ell_d \{f(n)\} \end{aligned}$$

$$= (e^s - 1) \ell_d \{f(n)\}(s) - f(1).$$

□

Next we define a discrete convolution operator on sequences which latter will be useful in solving discrete initial value problems.

DEFINITION 3. *Let $f, g : \mathbb{N} \to \mathbb{R}$ be two sequences . Then the discrete convolution of f and g denoted $(f * g)(n)$ is defined by*

$$(f * g)(n) := \sum_{k=1}^{n-1} f(k)\, g(n-k).$$

EXAMPLE 4. $(1 * 1) = n - 1$

EXAMPLE 5. $(n * 1) = \frac{n^2 - n}{2}$

EXAMPLE 6. $(n * n) = \frac{n^3 - n}{6}$

PROPOSITION 3. *(Transform of a discrete convolution).*

$$\ell_d \{(f * g)(n)\}(s) = \ell_d \{f(n)\}\, \ell_d \{g(n)\}.$$

PROOF. From the product of the two series:

$$\left(\sum_{n=1}^{\infty} a_n x^n\right)\left(\sum_{n=1}^{\infty} b_n x^n\right) = \sum_{n=2}^{\infty} c_n x^n$$

where $c_n = \sum_{k=1}^{n-1} a_k b_{n-k}$, we have,

$$\begin{aligned} \ell_d \{(f * g)(n)\}(s) &= \sum_{n=1}^{\infty} (f * g)(n)\, e^{-sn} \\ &= \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} f(k)\, g(n-k)\right) e^{-sn} \end{aligned}$$

$$\begin{aligned} &= \left(\sum_{n=1}^{\infty} f(n)\, e^{-sn}\right)\left(\sum_{n=1}^{\infty} g(n)\, e^{-sn}\right) \\ &= \ell_d \{f(n)\}\, \ell_d \{g(n)\}. \end{aligned}$$

□

COROLLARY 2. $\ell_d \left\{\sum_{k=1}^{n-1} f(k)\right\}(s) = \frac{s\ell_d\{f(n)\}}{e^s - 1}$.

PROOF. Follows from the fact that choosing $g \equiv 1$, we have

$$(f * g)(n) = f(n) * 1 = \sum_{k=1}^{n-1} f(k).$$

Then taking the transform of both sides, we have the result. □

PROPOSITION 4. *For a sequence $f : \mathbb{N} \to \mathbb{R}$,*

$$\ell_d \{nf(n)\}(s) = -\frac{d}{ds}\left(\ell_d \{f(n)\}(s)\right).$$

PROOF.

$$\frac{d}{ds}\left(\ell_d\left\{f(n)\right\}(s)\right) = \frac{d}{ds}\sum_{n=1}^{\infty} f(n)\, e^{-sn}$$

$$= \sum_{n=1}^{\infty}\left(-nf(n)\, e^{-sn}\right) = -\ell_d\left\{nf(n)\right\}(s).$$

□

COROLLARY 3. *For $k \in \mathbb{N}$, $\ell_d\left\{n^k f(n)\right\}(s) = (-1)^k \frac{d^k}{ds^k}\ell_d\left\{f(n)\right\}(s)$.*

REMARK 1. *By taking $f \equiv 1$, we get the relation:*

$$\begin{aligned} \ell_d\left\{n^k\right\}(s) &= (-1)^k \frac{d^k}{ds^k}\ell_d\left\{1\right\}(s) \\ &= (-1)^k \frac{d^k}{ds^k}\left(\frac{1}{e^s-1}\right). \end{aligned}$$

3. Initial value problems of discrete differential equations.

In this section we solve initial value problems of discrete differential equations using the discrete Laplace transform.

PROPOSITION 5. *The first order discrete IVP: $\triangle f(n) = n$, $f(1) = 1$, has solution given by*

$$f(n) = 1 + \frac{n^2-n}{2}.$$

PROOF. Taking the transform of both sides of the equation : $\ell_d\left\{\triangle f(n)\right\}(s) = \ell_d\left\{n\right\}$, we get

$$\left(e^s-1\right)\ell_d\left\{f(n)\right\}(s) - f(1) = \frac{e^s}{\left(e^s-1\right)^2}.$$

Substituting the value $f(1) = 1$, and simplifying the expression we get,

$$\ell_d\left\{f(n)\right\}(s) = \frac{1}{e^s-1} + \frac{e^s}{\left(e^s-1\right)^3}.$$

Then taking the inverse transform we have the solution :

$$\begin{aligned} f(n) &= \ell_d^{-1}\left\{\frac{1}{e^s-1} + \frac{e^s}{\left(e^s-1\right)^3}\right\} \\ &= 1 + (n * 1) = 1 + \frac{n^2-n}{2}. \end{aligned}$$

Here "$*$ " is the convolution operator. □

PROPOSITION 6. *The second order IVP: $\triangle^2 f(n) = n$, $f(1) = 1$, $\triangle f(1) = 2$, has solution given by :*

$$f(n) = 2n - 1 + \frac{n(n-1)(n-2)}{6}.$$

Proof. First, $\triangle^2 f(n) = \triangle(\triangle f(n)) = f(n+2) - 2f(n+1) + f(n)$ and using the initial conditions we get:

$$\ell_d\left\{\triangle^2 f(n)\right\}(s) = \left(e^{2s} - 2e^s + 1\right)\ell_d\{f(n)\}(s) - e^s - 1.$$

$$\Rightarrow (e^s-1)^2\,\ell_d\{f(n)\}(s) - e^s - 1 = \frac{e^s}{(e^s-1)^2}$$

$$\Rightarrow \ell_d\{f(n)\}(s) = \frac{1}{(e^s-1)^2} + \frac{e^s}{(e^s-1)^2} + \frac{e^s}{(e^s-1)^4}.$$

Then taking the inverse transform and using convolutions, we get the solution as :

$$\begin{aligned} f(n) &= (1*1) + n + \frac{n(n-1)(n-2)}{6} \\ &= 2n - 1 + \frac{n(n-1)(n-2)}{6}. \end{aligned}$$

□

Proposition 7. *$\ell_d\left\{\frac{1}{n}\right\}(s) = s - \ln(e^s - 1)$ for $s > 0$.*

Proof. $\ell_d\{1\}(s) = \frac{1}{e^s-1} = \sum_{n=1}^{\infty} e^{-sn}$ for $s > 0$. Integrating both sides, we get

$$\begin{aligned} \ln(e^s-1) - s &= \sum_{n=1}^{\infty}\left(-\frac{e^{sn}}{n}\right) \\ &= \ell_d\left\{\frac{-1}{n}\right\}(s) \end{aligned}$$

$$\Rightarrow \ell_d\left\{\frac{1}{n}\right\}(s) = s - \ln(e^s-1).$$

□

We now present discrete IVPs whose solutions are rational sequences in n.

Proposition 8. *The discrete IVP : $n\triangle f(n) = 1$, $f(2) = 2$, for $n \geq 2$ has solution given by: $f(n) = 1 + \sum_{k=1}^{n-1} \frac{1}{k}$.*

Proof. Taking the transform of both sides of the equation, we have:

$$\ell_d\{n\triangle f(n)\}(s) = \frac{1}{e^s-1}.$$

But

$$\ell_d\{n\triangle f(n)\}(s) = \ell_d\{nf(n+1) - nf(n)\}(s)$$

$$= e^s\ell_d\{nf(n)\}(s) - e^s\ell_d\{f(n)\}(s) - \ell_d\{nf(n)\}(s)$$

$$= (e^s-1)\,\ell_d\{nf(n)\}(s) - e^s\ell_d\{f(n)\}(s).$$

Thus,

$$\left(e^{s}-1\right)\ell_{d}\left\{nf\left(n\right)\right\}-e^{s}\ell_{d}\left\{f\left(n\right)\right\}=\frac{1}{e^{s}-1}.$$

Again,

$$\ell_{d}\left\{nf\left(n\right)\right\}\left(s\right)=-\frac{d}{ds}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right).$$

Therefore,

$$-\left(e^{s}-1\right)\frac{d}{ds}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)-e^{s}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)=\frac{1}{e^{s}-1}$$

which is an ordinary non-homogenious linear differential equation of first order in s. writing it in standard form we have :

$$\frac{d}{ds}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)+\frac{e^{s}}{e^{s}-1}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)=-\frac{1}{\left(e^{s}-1\right)^{2}}$$

whose solution for $\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)$ is given by $\frac{1}{e^{s}-1}-\frac{\ln(e^{s}-1)-s}{e^{s}-1}$. Then taking the inverse transform , we have the solution to be :

$$\begin{aligned}f\left(n\right) &= \ell_{d}^{-1}\left\{\frac{1}{e^{s}-1}+\frac{s-\ln\left(e^{s}-1\right)}{e^{s}-1}\right\}\\ &= 1+\ell_{d}^{-1}\left\{s-\ln\left(e^{s}-1\right)\right\}*\ell_{d}^{-1}\left\{\frac{1}{e^{s}-1}\right\}\end{aligned}$$

$$=1+\left(\frac{1}{n}*1\right)=1+\sum_{k=1}^{n-1}\frac{1}{k}\text{ for }n>1.$$

□

Proposition 9. *For $n\geq 2$, the IVP : $\triangle f\left(n\right)=\frac{1}{n^{2}}$, $f\left(2\right)=2$ has solution given by $f\left(n\right)=1+\sum_{k=1}^{n-1}\frac{1}{k^{2}}$.*

Proof. Re-writing the difference equation as : $n\triangle f\left(n\right)=\frac{1}{n}$, taking the transform of both sides and using corollary 3.3 we get

$$\frac{d}{ds}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)+\frac{e^{s}}{e^{s}-1}\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)=-\frac{s-\ln\left(e^{s}-1\right)}{e^{s}-1}.$$

Again solving for $\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)$, we have

$$\ell_{d}\left\{f\left(n\right)\right\}\left(s\right)=\frac{1}{e^{s}-1}-\frac{1}{e^{s}-1}\int\left(s-\ln\left(e^{s}-1\right)\right)ds.$$

$$\Rightarrow f\left(n\right)=1-\ell_{d}^{-1}\left\{\frac{1}{e^{s}-1}\right\}*\ell_{d}^{-1}\left\{\int\left(s-\ln\left(e^{s}-1\right)\right)ds\right\}$$

$$\begin{aligned}&= 1-\left(1*\left(-\frac{1}{n^{2}}\right)\right)\\ &= 1+\sum_{k=1}^{n-1}\frac{1}{k^{2}}.\end{aligned}$$

□

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